Advanced Calculus

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Examples using the Chain Rule

If $w = f(x, y, z)$, $x = \phi(t)$, $y = \psi(t)$, $z = \theta(t)$, find $\frac{dw}{dt}$.

Solution: $\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$.

Or: $\frac{dw}{dt} = \frac{\partial f}{\partial x}\phi'(t) + \frac{\partial f}{\partial y}\psi'(t) + \frac{\partial f}{\partial z}\theta'(t)$.

If $w = F(x, u, t)$, $u = f(x, t)$, $x = \phi(t)$, find $\frac{dw}{dt}$.

Solution: First substitute $f(x, t)$ for u, and $\phi(t)$ for x.

$$\begin{aligned} \frac{dw}{dt} &= \frac{d}{dt} F(\phi(t), f(\phi(t), t), t) \\ &= \frac{\partial F}{\partial x}\phi'(t) + \frac{\partial F}{\partial u}\frac{d}{dt} f(\phi(t), t) + \frac{\partial F}{\partial t} \\ &= \frac{\partial F}{\partial x}\phi'(t) + \frac{\partial F}{\partial u}\left(\frac{\partial f}{\partial x}\phi'(t) + \frac{\partial f}{\partial t}\right) + \frac{\partial F}{\partial t}. \end{aligned}$$

If $w = f(x, u, v)$, $u = f(x, y))$, $v = g(x, z)$, find $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial w}{\partial z}$.

Solution: First substitute $f(x, y)$ for u, and $g(x, z)$ for v.

$$\begin{aligned} \frac{dw}{dx} &= \frac{\partial}{\partial x} F(x, f(x, y), g(x, z)) \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial g}{\partial x}. \end{aligned}$$

$$\begin{aligned} \frac{dw}{dy} &= \frac{\partial}{\partial y} F(x, f(x, y), g(x, z)) \\ &= \frac{\partial F}{\partial u}\frac{\partial f}{\partial y}. \end{aligned}$$

If $w = f(x, y)$ and $y = F(x)$, find $\frac{dw}{dx}$ and $\frac{d^2w}{dx^2}$.

Solution: $\frac{dw}{dx} = \frac{d}{dx} f(x, F(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} F'(x)$.

$$\begin{aligned}\frac{d^2w}{dx^2} &= \frac{d}{dx}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} F'(x)\right), \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}\left(F'(x)\right)^2 + \frac{\partial f}{\partial y} F''(x).\end{aligned}$$

If x, y, and z are related by the equation: $x^2 + yz^2 + y^2x + 1 = 0$, find $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$ when $x = -1$ and $z = 1$.

Solution: Assume that there is a C^1 function $f(x,z)$ such that $y = f(x,z)$ satisfies the equation.

$$x^2 + f(x,z)z^2 + (f(x,z))^2 x + 1 = 0.$$

Then

$$\frac{\partial}{\partial x}\left(x^2 + f(x,z)z^2 + (f(x,z))^2 x + 1\right)$$
$$= 2x + \frac{\partial f}{\partial x}(x,z)z^2 + 2f(x,z)\frac{\partial f}{\partial x}(x,z) = 0$$

Solve for $\frac{\partial f}{\partial x}$:

$$\frac{\partial f}{\partial x} = -\frac{2x}{z^2 + 2f(x,z)}.$$

We need to solve for $y = f(x,z)$ when $x = -1$ and $z = 1$, in order to determine $\frac{\partial f}{\partial x}$:

$$1 + y - y^2 + 1 = 0; \qquad y = 2, -1.$$

If $y = -1$, $\frac{\partial f}{\partial x} = \frac{2}{1-2} = -2$. If $y = 2$, $\frac{\partial f}{\partial x} = \frac{2}{1+4} = \frac{2}{5}$.

Let x, y, u, v be related by the equations:

$$xy + x^2 u = vy^2$$
$$3x - 4uy = x^2 v$$

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$. *Solution:* Let $f(x, y, u, v) = xy + x^2 u - vy^2$, and $g(x, y, u, v) = 3x - 4uy - x^2 v$. Assume that these equations determine that $u = \alpha(x, y)$, $v = \beta(x, y)$. Compute:

$$\frac{\partial}{\partial x} f(x, y, \alpha(x, y), \beta(x, y))$$
$$= y + 2x\alpha(x, y) + x^2 \frac{\partial \alpha}{\partial x}(x, y) - y^2 \frac{\partial \beta}{\partial x}(x, y) = 0.$$

$$\frac{\partial}{\partial x} g(x, y, \alpha(x, y), \beta(x, y))$$
$$= 3 - 4y\frac{\partial \alpha}{\partial x}(x, y) - 2x\beta(x, y) - x^2\frac{\partial \beta}{\partial x}(x, y) = 0.$$

Organize as a linear system in the unknowns $\frac{\partial \alpha}{\partial x}$, $\frac{\partial \beta}{\partial x}$:

$$x^2\frac{\partial \alpha}{\partial x}(x, y) - y^2\frac{\partial \beta}{\partial x}(x, y) = -y - 2x\alpha(x, y)$$
$$4y\frac{\partial \alpha}{\partial x}(x, y) + x^2\frac{\partial \beta}{\partial x}(x, y) = 3 - 2x\beta(x, y)$$

Write in matrix form:

$$\begin{bmatrix} x^2 & -y^2 \\ 4y & x^2 \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha}{\partial x} \\ \frac{\partial \beta}{\partial x} \end{bmatrix} = \begin{bmatrix} -y - 2x\alpha(x,y) \\ 3 - 2x\beta(x,y) \end{bmatrix}$$

Repace α with u and β with v, and apply Cramer's rule:

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -y - 2xu & -y^2 \\ 3 - 2xv & x^2 \end{vmatrix}}{\begin{vmatrix} x^2 & -y^2 \\ 4y & x^2 \end{vmatrix}} = \frac{-x^2y - 2x^3u + 3y^2 - 2xy^2v}{x^4 + 4y^3}$$