Advanced Calculus

Professor David Wagner

[1]Department of Mathematics
University of Houston

August 19

Who am I, contact info

Professor David Wagner
Office 615 PGH, Phone 713-743-3460
Course web page https://www.math.uh.edu/~wagner/3334/math_3334_fall_2019.html
Start with https://www.math.uh.edu/~wagner, click on link for Teaching, then click on link for 3334.

The set of real numbers $\mathbb{R}$ is a complete ordered field. It is the only one.

Properties:

1. Addition and multiplication are associative and commutative.
2. There is an additive identity, "1", and a multiplicative identity, "0".
3. Every real number x has an additive inverse, which is a real number $-x$, such that $x + (-x) = 0$.
4. Every non-zero real number x has a multplicative inverse, which is a real number $\frac{1}{x}$, such that $x * (\frac{1}{x}) = 1$.

Properties 1 - 4 characterize $\mathbb{R}$ as a field. Other fields include $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_p$—the rational numbers, the complex numbers, and the integers mod p, when p is a prime number.

Order properties of $\mathbb{R}$

1. For every pair of real numbers x and y, exactly one of the following is true:

$$x = y, \; x > y, \; x < y.$$

2. If $x < y$ and $y < z$, then $x < z$, (Transitive)
3. If $x < y$, then $x + z < y + z$ for any $z \in \mathbb{R}$.
4. If $x < y$ and $z > 0$, then $xz < yz$.

Have we left something out?
There are two ordered fields, $\mathbb{Q}$ and $\mathbb{R}$.

Let's try to prove this:

If $x < y$ and $z < 0$, then $xz > yz$.

Multiplication of an inequality by a negative number reverses the inequality.

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. □

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. □

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. □

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. □

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. $\square$

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. $\square$

Proof

- If $x < y$ and $z < 0$,
- Then $0 - z > 0$,
- Why?
- (If $z < 0$, then $0 = -z + z < -z + 0 = 0 - z$.)
- so that $(0 - z)x < (0 - z)y$,
- and so $-xz < -yz$.
- Now add $xz + yz$ to both sides to get:
- $-xz + xz + yz < -yz + xz + yz$, or
- $yz < xz$. $\square$

Completeness

We assume that $\mathbb{R}$ has the *least upper bound property*,. This means that any set S of real numbers which has an upper bound has a least upper bound y, which may or may not belong to S. It follows that any set T of real numbers which is bounded below has a greatest lower bound.

Completeness

Some definitions and properties:

- We say that a real number x is an upper bound for a subset S of $\mathbb{R}$, if $y \leq x$ for all $y \in S$.
- We say that a real number x is a lower bound for a subset S of $\mathbb{R}$, if $y \geq x$ for all $y \in S$.
- We say that a real number x is a least upper bound for a subset S of $\mathbb{R}$, if x is
  - an upper bound for S, and
  - a lower bound for the set of upper bounds for S.
- A real number x is a least upper bound for a subset S of $\mathbb{R}$, if an only if, for every $\epsilon > 0$, there is an upper bound y for S such that $x < y < x + \epsilon$. (Exercise!)

Counting

We say that a set S is

- *Countable* if S is not finite (infinite) and S can be put in 1-1 correspondence with the natural numbers $\mathbb{N}$.
- *Uncountable* if S is infinite and not countable.

The set of all rational numbers is countable. The set of real numbers is uncountable. To see why this matters, consider a function f on $\mathbb{R}$ for which $f(x) > 0$ for all $x \in \mathbb{R}$. Then $\sum_{x \in \mathbb{R}} f(x) = \infty$. But many countable sums (series) converge absolutely to a finite value.

Our old friend $\mathbb{R}^n$

You probably last saw $\mathbb{R}^n$ in linear algebra class. It is the first, and most important, example of a vector space that we encounter.
What we need to know from linear algebra:

- $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) =$ vector sum.
- $c\,(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) =$ scalar multiplication.
- $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1y_1 + \ldots + x_ny_n =$ the dot product.
- If $X = (x_1, \ldots, x_n)$, $\|X\| = \sqrt{X \cdot X} =$ the *norm* of X.
- $|X \cdot Y| \leq \|X\| \, \|Y\|$, the Cauchy-Schwartz inequality.

Triangle Inequality

A norm is supposed to satisfy the *Triangle Inequality*. Let's see if this norm does.

- $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + 2X\cdot Y + Y\cdot Y,$
- $\quad = \|X\|^2 + 2X\cdot Y + \|Y\|^2,$
- $\quad \leq \|X\|^2 + 2\|X\|\,\|Y\| + \|Y\|^2,$
- $\quad = (\|X\| + \|Y\|)^2.$
- Take the square root of both sides, and we get $\|X+Y\| \leq \|X\| + \|Y\|$, the Triangle Inequality.

Triangle Inequality

A norm is supposed to satisfy the *Triangle Inequality*. Let's see if this norm does.

- $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + 2X\cdot Y + Y\cdot Y,$
- $\quad = \|X\|^2 + 2X\cdot Y + \|Y\|^2,$
- $\quad \leq \|X\|^2 + 2\|X\|\,\|Y\| + \|Y\|^2,$
- $\quad = (\|X\| + \|Y\|)^2.$
- Take the square root of both sides, and we get $\|X+Y\| \leq \|X\| + \|Y\|$, the Triangle Inequality.

Triangle Inequality

A norm is supposed to satisfy the *Triangle Inequality*. Let's see if this norm does.

- $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + 2X\cdot Y + Y\cdot Y,$
- $\quad = \|X\|^2 + 2X\cdot Y + \|Y\|^2,$
- $\quad \leq \|X\|^2 + 2\|X\|\,\|Y\| + \|Y\|^2,$
- $\quad = (\|X\| + \|Y\|)^2.$
- Take the square root of both sides, and we get $\|X+Y\| \leq \|X\| + \|Y\|$, the Triangle Inequality.

Triangle Inequality

A norm is supposed to satisfy the *Triangle Inequality*. Let's see if this norm does.

- $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + 2X\cdot Y + Y\cdot Y,$
- $\quad = \|X\|^2 + 2X\cdot Y + \|Y\|^2,$
- $\quad \leq \|X\|^2 + 2\,\|X\|\,\|Y\| + \|Y\|^2,$
- $\quad = (\|X\| + \|Y\|)^2.$
- Take the square root of both sides, and we get $\|X+Y\| \leq \|X\| + \|Y\|$, the Triangle Inequality.

Triangle Inequality

A norm is supposed to satisfy the *Triangle Inequality*. Let's see if this norm does.

- $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X\cdot X + 2X\cdot Y + Y\cdot Y,$
- $\quad = \|X\|^2 + 2X\cdot Y + \|Y\|^2,$
- $\quad \leq \|X\|^2 + 2\|X\|\,\|Y\| + \|Y\|^2,$
- $\quad = (\|X\| + \|Y\|)^2.$
- Take the square root of both sides, and we get $\|X+Y\| \leq \|X\| + \|Y\|$, the Triangle Inequality.

Angles

The Cauchy-Schwartz inequality also gives us $\left|\frac{X}{\|X\|} \cdot \frac{Y}{|Y|}\right| \leq 1$. We get equality if and only if X and Y are parallel. This suggests that we identify $\frac{X}{\|X\|} \cdot \frac{Y}{|Y|}$ with $\cos(\theta)$ where θ is the angle between X and Y.

The most important angle for linear algebra or multi-variable calculus is the right angle, $\frac{\pi}{2}$, which has a cosine of 0. So we say that vectors X and Y in $\mathbb{R}^n$ are orthogonal (a fancy word for perpendicular) if $X \cdot Y = 0$.

Hyperplanes

You may recall from Calculus III, that a plane through (x_0, y_0, z_0) with normal vector $(A, B, C) \neq 0$ has equation

$$\begin{aligned}(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) \\ = A\,(x - x_0) + B\,(y - y_0) + C\,(z - z_0) \\ = 0.\end{aligned}$$

This plane is a surface with dimension 2 in $\mathbb{R}^3$. Since, at any point of the surface, the set of vectors that are normal (perpendicular) to the surface has dimension 1, we say that the surface has co-dimension 1.

More Hyperplanes

Similarly, in $\mathbb{R}^n$, the equation

$$(A_1, \ldots, A_n) \cdot (x_1 - x_{10}, \ldots, x_n - x_{n0})$$
$$= A_1 (x - x_{10}) + \ldots + A_n (x_n - x_{n0}) = 0.$$

describes a flat subset of $\mathbb{R}^n$ with dimension $n - 1$, co-dimension 1, through $(x_{10}, \ldots, x_{n0})$, and with normal vector $(A_1, \ldots, A_n)$, which we assume is not 0. We call this set "hyperplane". If we use the word "plane" to describe a subset of $\mathbb{R}^n$ with $n > 3$, we should mean a flat set with dimension 2.

To determine the dimension of a hyperplane, let $y_i = x_i - x_{i0}$, $i = 1, \ldots, n$. Then we have

$$A_1 y_1 + \ldots + A_n y_n = 0.$$

This is a linear homogeneous equation for which the matrix $\begin{pmatrix} A_1 \ldots A_n \end{pmatrix}$ has rank 1, so that the null space has dimension $n - 1$.

Lines

Recall from Calculus III that there are several ways to describe a line in $\mathbb{R}^3$ with equations:

- Scalar Parametric Equations: $x = x_0 + v_1 t$, $y = y_0 + v_2 t$, $z = z_0 + v_3 t$, $-\infty < t < \infty$.
- Vector Parametric Equations: $(x, y, z) = (x_0, y_0, z_0) + (v_1, v_2, v_3)\, t$, $-\infty < t < \infty$.
- Symmetric Equations: $\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$, when no $v_i = 0$.

Lines

Recall from Calculus III that there are several ways to describe a line in $\mathbb{R}^3$ with equations:

- Scalar Parametric Equations: $x = x_0 + v_1 t$, $y = y_0 + v_2 t$, $z = z_0 + v_3 t$, $-\infty < t < \infty$.
- Vector Parametric Equations: $(x, y, z) = (x_0, y_0, z_0) + (v_1, v_2, v_3)\, t$, $-\infty < t < \infty$.
- Symmetric Equations: $\frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$, when no $v_i = 0$.

Lines

Recall from Calculus III that there are several ways to describe a line in $\mathbb{R}^3$ with equations:

- Scalar Parametric Equations: $x = x_0 + v_1 t$, $y = y_0 + v_2 t$, $z = z_0 + v_3 t$, $-\infty < t < \infty$.
- Vector Parametric Equations: $(x, y, z) = (x_0, y_0, z_0) + (v_1, v_2, v_3)\, t$, $-\infty < t < \infty$.
- Symmetric Equations: $\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$, when no $v_i = 0$.

Lines in $\mathbb{R}^n$

Similarly, in $\mathbb{R}^n$ we have:

- Scalar Parametric Equations: $x_i = x_{i0} + v_i t,\ i = 1,\ \ldots,\ n,$ $-\infty < t < \infty.$
- Vector Parametric Equations: $(x_1, \ldots, x_n) = (x_{10}, \ldots, x_{n0}) + (v_1, \ldots, v_n)\, t,\ -\infty < t < \infty,$ or $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t.$
- Symmetric Equations: $\frac{x_i - x_{i0}}{v_i}$ is independent of i.

For lines in $\mathbb{R}^n$ we usually stick with Scalar or Vector Parametric equations.

Lines in $\mathbb{R}^n$

Similarly, in $\mathbb{R}^n$ we have:

- Scalar Parametric Equations: $x_i = x_{i0} + v_i t$, $i = 1, \ldots, n$, $-\infty < t < \infty$.
- Vector Parametric Equations: $(x_1, \ldots, x_n) = (x_{10}, \ldots, x_{n0}) + (v_1, \ldots, v_n)\, t$, $-\infty < t < \infty$, or $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}t$.
- Symmetric Equations: $\frac{x_i - x_{i0}}{v_i}$ is independent of i.

For lines in $\mathbb{R}^n$ we usually stick with Scalar or Vector Parametric equations.

Line segments

If p and q are distinct points in $\mathbb{R}^n$, we can use $q - p$ as a direction vector for the line though p and q: $\mathbf{x} = p + \lambda(q - p)$, $-\infty < \lambda < \infty$. Then $0 \leq \lambda \leq 1$ corresponds to the closed line segment from p to q, which we denote as $[p, q]$.

Convexity

We say that a subset S of $\mathbb{R}^n$ is *convex*, if for every pair p, q of points in S, the closed line segment $[p, q]$ is contained in S. Convexity is one of the most important concepts in mathematical analysis.

Open ball

The *open ball* in $\mathbb{R}^n$ with radius $r > 0$ and center $x_0 \in \mathbb{R}^n$ is

$$B(x_0, r) = \{x \in \mathbb{R}^n : \ \|x - x_0\| < r\}.$$

We show that $B(0, r)$ is convex. Suppose $p, \ q \in \mathbb{R}^n$ so that $\|p\|, \ \|q\| < r$. Then for $0 \leq lambda \leq 1$,

$$\|(1-\lambda)p + \lambda q\| \leq (1-\lambda)\|p\| + lambda\|q\| < (1-\lambda)r + lambdar = r.$$

Here we have used the Triangle Inequality and the homogeneity of the norm.
Thus every point of the line segment $[p.q]$ is an element of $B(0, r)$.
So $B(0, r)$ is convex.