Advanced Calculus

Professor David Wagner

[1]Department of Mathematics
University of Houston

August 21

Who am I, contact info

Professor David Wagner
Office 615 PGH, Phone 713-743-3460
Course web page https://www.math.uh.edu/~wagner/3334/math_3334_fall_2019.html
Start with https://www.math.uh.edu/~wagner, click on link for Teaching, then click on link for 3334.

Definition of function

Our main objects of study are *functions*.

Definition

Let n and m be natural numbers, and let Ω be a subset of $\mathbb{R}^n$. A function from Ω to $\mathbb{R}^m$ is a rule that assigns to each $x \in \Omega$ a unique point $f(x) \in \mathbb{R}^m$.

Examples

Example

A curve in $\mathbb{R}^m$ can be parameterized by a function $f : (a, b) \to \mathbb{R}^m$.

Example

This function maps $\mathbb{R}^2$ to itself: $f(x, y) = \left(x^2 - y^2,\ 2xy\right)$. This is the same mapping of $\mathbb{R}^2$ that is given in complex form as $F(z) = z^2$.

Example

We can map a region in $\mathbb{R}^2$ into $\mathbb{R}^3$ to parameterize a surface:

$$x = \phi(u, v),\ \ y = \psi(u, v),\ \ z = \theta(u, v), \quad (u, v) \in \Omega.$$

Another example

Example

A mathematically perfect mountain might have an elevation function given by

$$f(x,y) = \begin{cases} 30,000 - x^2 - y^2, & x^2 + y^2 \leq 30,000 \\ 0, & \textit{otherwise} \end{cases}$$

But perfection is in the eye of the beholder. Note that the graph of this function is also a surface.

Surfaces as level sets

As we saw last class, a surface can occur as a level set of a function on $\mathbb{R}^3$.

Example

If $f(x, y, z) = x^2 + y^2 + z^2$, what are the level sets of f?

Graph of a function

Definition

Let Ω be a subset of $\mathbb{R}^n$, and let f be a function from Ω to $\mathbb{R}^m$. The em graph of f is the set $\{(x, f(x)) : \ x \in \Omega\}$. It is a subset of $\Omega \times \mathbb{R}^m$ which is a subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Example

For example, if $f(x) = x$ for all $x \in \Omega$, then $f : \Omega \to \Omega$ and the graph of f is $\{(x, x) : x \in \Omega\}$.

When $n + m > 3$ it is difficult to visualize the graph of a function which maps $\Omega \subset \mathbb{R}^n \to \mathbb{R}^m$.

Convexity

Definition

Let Ω be a convex subset of $\mathbb{R}^n$. A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f\left((1-\lambda)p + \lambda q\right) \leq (1-\lambda)f(p) + \lambda f(q)$$

for all p and q in Ω and $0 \leq \lambda \leq 1$. f is said to be *midpoint convex* if this relationship holds for $\lambda = \frac{1}{2}$.

One can prove that a function f which is midpoint convex and continuous on Ω must be convex on Ω.

Examples of convex functions

- $f(x) = e^x$
- $f(x) = e^{-2x}$
- $f(x, y) = x^2 + y^2$
- $f(x) = |x|$
- $f(x, y) = \|(x, y)\| = \sqrt{x^2 + y^2}$

Sequences

A sequence in $\mathbb{R}^n$ is a function with domain $\mathbb{N} = \mathbb{Z}^+$ and with values in $\mathbb{R}^n$. In Calculus II we study sequences in $\mathbb{R}$. We normally use notation like "a_n" to denote the value of the sequence at n.

Examples:

- $f(n) = a_n = \frac{1}{n}$
- $g(n) = (x_n, y_n) = \left(2 + \frac{3}{n^2}, 5 + e^{-n}\right)$
- $h(n) = c_n = \sin\left(\frac{1}{n}\right)$.

Sequences of functions

An important topic in analysis is sequences and series of functions. For example:

$$f_n(x) = nx^2 e^{-nx}.$$

For such a sequence, we would like to know if there is a function $f(x)$ to which $f_n(x)$ converges, and how it converges. Under what conditions is the limit function continuous, or differentiable? Is

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx?$$

Definitions

- A point x is said to be an *interior point* of a subset S of $\mathbb{R}^n$ if for some (sufficiently small) $r > 0$, $B(x, r) \subset S$.
- The *interior* of a set S is the set of all interior points of S.
- A set S is said to be *open* if every point in S is an interior point of S. Thus S is open if and only if S is equal to its interior.

Open balls are open!

Theorem

Every open ball $B(x, r)$ with $r > 0$ and $x \in \mathbb{R}^n$ is an open set.

Proof.

To make this simple, assume $x = 0$. Draw a picture of the ball to help visualize the proof.

- If $p \in B(0, r)$, then $\|p\| < r$.
- Let $\delta = r - \|p\| > 0$.
- If $q \in B(p, \delta)$, then $\|q - p\| < \delta$.
- Then $\|q\| = \|q - p + p\| \leq \|q - p\| + \|p\|$ by the Triangle Inequality.
- $\|q - p\| + \|p\| < \delta + \|p\| = r$.
- So if $q \in B(p, \delta)$, then $\|q\| < r$.

□

Open balls are open!

- So $B(p, \delta) \subset B(0, r)$.
- Thus $B(0, r)$ is open.

Corollary

Open intervals (a, b) are open subsets of $\mathbb{R}$.

Proof.

Let $\delta = \frac{b-a}{2}$, $c = \frac{a+b}{2}$. Then $(a, b) = B(c, \delta)$. Thus (a, b) is an open ball. □

More Definitions

- A point p is an *exterior point* of a set S, if for some (small) $r > 0$, $B(p, r) \cap S = \phi$. This means that p is an interior point of S^c.
- A set S is said to be *closed* if S^c is open.
- A point p is said to be a *boundary point* of a set S if p is neither an interior point nor an exterior point of S. This means that every ball $B(p, r)$ contains at least one point of S and at least one point of S^c.

Theorem

- *A set S is closed if and only if S contains all of its boundary points.*
- *A set S is open if and only if S contains none of its boundary points.*

Proof.

Exercise! □

More Definitions

- The set of all boundary points of a set S is called the *boundary* of S. (duh!) It is denoted by $bdy(S)$.
- The *closure* of a set S is formed by adjoining to S all of its boundary points. It is denoted by $\overline{S}$. Thus

$$\bar{S} = S \cup bdy(S).$$

- For example, $\overline{B(p,r)} = \{x : \ \|x - p\| \leq r\}$ = the closed ball centered at p with radius r.
- A set S is said to be *bounded* if for some M, $S \subset B(0, M)$. This has nothing to do with the boundary of S.

Even More Definitions

- A set U is a *neighborhood* of a point p if p is an interior point of U.

An example

Example

Let

$$A = \left\{ p \in \mathbb{R}^2 : \; 0 < \|p\| \leq 1 \right\} \cup \{(0,2)\}.$$

- The boundary of A is the union of the circle $\|p\| = 1$ with the points $\{(0,0), \; (2,0)\}$.
- The interior of A is the set where $0 < \|p\| \; < 1$.
- The closure of A is the union of A with $\{(0,0)\}$.

More definitions

- A point p is said to be a *cluster point* of a set S if every neighborhood of p contains infinitely many points of S.
- A point p is said to be an *isolated point* of a set S if $p \in S$ and there is a neighborhood U of p such that $U \cup S = \{p\}$.

In our example, $(0, 0)$ is a cluster point of A and $(2, 0)$ is an isolated point of A.

Basic Theorems

1. If A and B are open sets, so are $A \cup B$ and $A \cap B$.
2. The union of any collection of open sets is open, but the intersection of an infinite collection of open sets *need not be open!*
3. If A and B are closed sets, so are $A \cup B$ and $A \cap B$.
4. The intersection of any collection of closed sets is closed, but the union of an infinite collection of closed sets *need not be closed!*
5. A set is open if and only if its complement is closed.
6. The interior of a set S is the largest open subset of S.
7. The closure of a set S is the smallest closed set that contains S.

More Basic Theorems