Advanced Calculus

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August 26

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Basic Theorems

1. If A and B are open sets, so are $A \cup B$ and $A \cap B$.
2. The union of any collection of open sets is open, but the intersection of an infinite collection of open sets *need not be open!*
3. If A and B are closed sets, so are $A \cup B$ and $A \cap B$.
4. The intersection of any collection of closed sets is closed, but the union of an infinite collection of closed sets *need not be closed!*
5. A set is open if and only if its complement is closed.
6. The interior of a set S is the largest open subset of S.
7. The closure of a set S is the smallest closed set that contains S.

More Basic Theorems

1. The boundary of a set S is always a closed set and is the intersection of $\bar{S}$ and $\overline{S^c}$.
2. A set S is closed if and only if every cluster point for S belongs to S.
3. The interior of S is obtained by deleting every point in S that is on the boundary of S.

Definitions: Connected sets

Definition

Two non-empty disjoint sets A and B are said to be *mutually separated* if neither contains a boundary point of the other. A set is *disconnected* if it is the union of separated subsets. A set is *connected* if it is not disconnected.

Examples

- $B(p, r)$ is connected for every $p \in \mathbb{R}^n$ and every $r > 0$.
- In $\mathbb{R}^n$, let $\mathbf{0} = (0, \ldots, 0)$ and let $p = (2, 0, \ldots, 0)$. Then $A = B(\mathbf{0}, 1) \cup B(p, 1)$ is disconnected—because $B(\mathbf{0}, 1)$ and $B(p, 1)$ are mutually separated.

Polygon connected sets

It is easier to work with this type of connectedness:

Definition

A set S is said to be *polygon connected* if, given any two points [and q in S, there is a chain of line segments in S which abut and form a path that starts at p and ends at q.

Example

Every convex set is polygon connected.

Sequences

- A sequence in $\mathbb{R}^n$ is a function from $\mathbb{Z}^+ = \{1,\ 2,\ 3,\ \ldots\}$ to $\mathbb{R}^n$. We might write $f(n) = p_n = \ldots$. We might refer to the sequence as $\{p_n\}$.
- The *trace* of a sequence $\{p_n\}$ is the set of values of p_n. Thus, if $p_n = (-1)^n$, then the trace of $\{p_n\}$ is the set $\{1,\ -1\}$.
- A sequence is *bounded* if its trace is a bounded set. To put it directly, $\{p_n\}$ is bounded if there is a number M such that $\|p_n\| \leq M$ for all $n \in \mathbb{Z}^+$.

Limits of sequences

Definition

A sequence $\{p_n\}$ *converges* to the point p if for every neighborhood U of p, there is a number N such that $p_n \in U$ whenever $n \geq N$.

To say that $\{p_n\}$ converges to p, we often write;

$$\lim_{n\to\infty} p_n = p, \text{ or simply } p_n \to p.$$

Facts about convergence

Note:

- $\|p_n - p\|$ does not need to decrease monotonically for $p_n \to p$.
- n is never equal to ∞.
- A sequence $\{p_n\}$ is said to be *convergent* if there is a point p to which it converges.
- A sequence that is not convergent is said to be *divergent*.
- Examples of divergent sequences:

$$a_n = n^2, \quad b_n = (-1)^n\, n, \quad c_n = 1+(-1)^n, \quad d_n = \left((-1)^n, \frac{1}{n}\right).$$

Convergence and Tails

Convergence depends only on the "tail" of a sequence: If $p_n = q_n$ for $n \geq N$, and $\{q_n\}$ converges, then p_n converges (to the same limit).

An example proof of convergence

Example

Let $p_n = \left(\frac{1}{n},\ \frac{n-1}{n}\right) \in \mathbb{R}^2$. A little bird tells us that $p_n \to (0,1)$. Let's prove it.

- Let U be a neighborhood of $(0,1)$.
- Then for some $\epsilon > 0$, $B(p,\epsilon) \subset U$.
- We need to find a condition on N so that $\|p_n - (0,1)\| < \epsilon$ whenever $n \geq N$.
- Calculate $\|p_n - (0,1)\| = \left\|\left(\frac{1}{n}, -\frac{1}{n}\right)\right\| = \sqrt{\frac{2}{n^2}}$
- $= \frac{\sqrt{2}}{n}$.
- Solve the inequality $\frac{\sqrt{2}}{n} < \epsilon$: $n > \frac{\sqrt{2}}{\epsilon}$. So if $N = \frac{\sqrt{2}}{\epsilon} + 1$, then $\|p_n - (0,1)\| < \epsilon$ whenever $n \geq N$.

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Theorems on Convergent Sequences

- Theorem: Every convergent sequence is bounded.
- Theorem: If $p_n \to p \in \mathbb{R}^n$ and $q_n \to q \in \mathbb{R}^n$ then $p_n + q_n \to p + q$.
- Theorem: The closure of a set S in R^n is the set of all limits of converging sequences of points in S.
- Corollary: Every point in the boundary of a set $S \in \mathbb{R}^n$ is simultaneously the limit of a converging sequence in S and of a converging sequence in S^c.
- Corollary: A set is closed if and only if it contains the limit of every converging sequence $\{p_n\}$ in S.

Limit points

Definition

Let $\{p_n\}$ be a sequence in $\mathbb{R}^k$. We say that $p \in \mathbb{R}^k$ is a *limit point* of $\{p_n\}$, if for every neighborhood U of p, there is an infinite set $A \subset \mathbb{Z}^+$ such that $p_n \in U$ for all $n \in A$.

Example

The sequence $p_n = (-1)^n$ diverges but it has two limit points in $\mathbb{R}$: 1 and -1. Note that $\{p_n\}$ has two *subsequences*, $q_n = p_{2n}$ and $r_n = p_{2n+1}$, such that $q_n \to 1$ and $r_n \to -1$.

Definition

We say that a sequence $\{q_n\}$ is a *subsequence* of a sequence $\{p_n\}$ if there is an increasing sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that $q_k = p_{n_k}$, $k \in \mathbb{Z}^+$.

Limit points and subsequences, vector limits

Theorem

A point p is a limit point of a sequence $\{p_n\}$ if and only if there is a subsequence $\{q_n\}$ of $\{p_n\}$ such that $q_n \to p$.

Theorem

Let $\{p_n\}$ be a sequence in $\mathbb{R}^k$ with $p_n = (x_{1,n}, \ldots, x_{k,n})$. Let $p = (x_1, \ldots, x_k)$. Then $p_n \to p$ if and only if $x_{j,n} \to x_j$ for $j = 1, \ldots, k$.

Limit theorems

Theorem

If $a_n \to 0 \in \mathbb{R}$ and $\{b_n\} \in \mathbb{R}$ is bounded, then $a_n b_n \to 0$.

Corollary

If $a_n \to A$, then for any real number c, $ca_n \to cA$.

Theorem

If $a_n \to A \in \mathbb{R}$ and $b_n \to B \in \mathbb{R}$, then $a_n b_n \to AB$.

Proof.

$$a_n b_n = (a_n - A)\, b_n + A b_n \to 0 + AB = AB.$$

□

More Limit theorems

Lemma

Suppose $a_n \to A \neq 0 \in \mathbb{R}$. Then there is an n_0 such that $\frac{1}{|a_n|} < \frac{2}{|A|}$ for all $n > n_0$; thus, the sequence $\left\{\frac{1}{a_n}\right\}$ is defined and bounded for all $n > n_0$.

Corollary

If $a_n \to A \neq 0 \in \mathbb{R}$, then $\frac{1}{a_n} \to \frac{1}{A}$.

Proof.

Choose n_0 as in the Lemma. Then for $n > n_0$,

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \left|\frac{A - a_n}{a_n A}\right| \leq \left|\frac{2\,(A - a_n)}{A^2}\right| \to 0,$$

by a previous Corollary. □

Yet another Limit Theorem

Theorem

If $a_n \to A \in \mathbb{R}$ *and* $b_n \to B \neq 0 \in \mathbb{R}$*, then* $\frac{a_n}{b_n} \to \frac{A}{B}$.