Advanced Calculus

Professor David Wagner

[1]Department of Mathematics
University of Houston

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Examples

Example

$$\lim_{k\to\infty} \frac{\log k}{k} = 0.$$

Given k, choose m so that $(m-1)^2 \leq k < m^2$. Check that $m^2 < 2^{m-1}$ if $m \geq 7$. With $m = 7$ we have $6^2 \leq k < 7^2$. Then for $k \geq 36$ we have (take the log of $k < m^2 < 2^{m-1}$):

$$\log k < \log\left(m^2\right) < \log\left(2^{m-1}\right) = (m-1)\log 2 \leq \sqrt{k}\log 2.$$

Then

$$\frac{\log k}{k} < \frac{\log 2}{\sqrt{k}} \to 0.$$

Theorem

For $b > 1$ and any r,

$$\lim_{k\to\infty} \frac{k^r}{b^k} = 0.$$

Proof.

Since $\frac{\log k}{k} \to 0$, there is N such that $\frac{\log k}{k} < \frac{\log b}{r+1}$ for $k > N$. Then $(1+r)\log k < k \log b$, or $k^{r+1} < b^k$. So $\frac{k^r}{b^k} < \frac{1}{k} \to 0$. □

Corollary

If $0 < a < 1$, $a^k \to 0$.

The Monotone Sequence Propwrty

Definition

A sequence $\{a_k\}$ of real numbers is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{Z}^+$, and it is *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbb{Z}^+$. A sequence of real numbers is *monotonic* if it is either increasing or decreasing.

We will take the following statement, called *The Monotonic Sequence Propeerty* as an axiom:

> *Every bounded monotonic sequence of real numbers is convergent.*

Examples

Example

Let $b_n = \left(1+\frac{1}{n}\right)^{n+1}$. Then $\{b_n\}$ is decreasing and bounded, so it converges (to e).

$$\begin{aligned}\frac{b_n}{b_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} &= \left(\frac{n+1}{n}\right)^{n+1}\left(\frac{n+1}{n+2}\right)^{n+2} \\ &= \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+1}\left(\frac{n+1}{n+2}\right) \\ &= \left(1+\frac{1}{n^2+2n}\right)^{n+1}\left(\frac{n+1}{n+2}\right)\end{aligned}$$

Lemma to finish

Lemma

For any integer $m > 0$ and any $x > 0$,

$$(1 + x)^m > 1 + mx.$$

Proof.

By the binomial theorem,
$(1 + x)^m = 1 + mx + \text{positive terms} > 1 + mx$ □

Then

$$\left(1 + \frac{1}{n^2 + 2n}\right)^{n+1} > 1 + (n + 1)\left(\frac{1}{n^2 + 2n}\right) = \frac{n^2 + 3n + 1}{n^2 + 2n}.$$

Finish

Then

$$\frac{b_n}{b_{n+1}} > \left(\frac{n^2+3n+1}{n^2+2n}\right)\left(\frac{n+1}{n+2}\right) = \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} > 1.$$

So $b_n > b_{n+1}$, thus $\{b_n\}$ is decreasing. Since $4 \geq b_n > 0$, $\{b_n\}$ is also bounded. Thus, by the Monotonic Sequence Property, b_n converges to some L, $0 \leq L \leq 4$. If $a_n = \left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^{-1} b_n$, a_n converges to the same limit.

Extracting square roots

Theorem

Let A and x_1 be any positive numbers. Let $x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x}\right)$, $n = 1,\ 2,\ 3,\ \ldots$. Then x_n converges to $\sqrt{A}$.

Proof.

Assume $x_1 \neq \sqrt{A}$. We show that after x_2, the sequence is decreasing.

$$\begin{aligned}(x_2)^2 - A &= \frac{1}{4}\left(x_1^2 + 2A + \frac{A^2}{x_1^2}\right) - A \\ &= \frac{1}{4}\left(x_1^2 - 2A + \frac{A^2}{x_1^2}\right) = \frac{1}{4}\left(x_1 - \frac{A}{x_1}\right) > 0.\end{aligned}$$

So for any non-zero choice of x_1, $x_2 > \sqrt{A}$. □

More extraction

Similarly, $x_n > \sqrt{A}$ for all $n \geq 2$. The sequence is decreasing because

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) = \frac{1}{2}\left(x_n - \frac{A}{x_n}\right) = \frac{x_n^2 - A}{2x_n} > 0.$$

Since

$$x_{n+1} - \sqrt{A} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) - \sqrt{A} = \frac{\left(x_n - \sqrt{A}\right)^2}{2x_n}$$

$$< \frac{\left(x_n - \sqrt{A}\right)^2}{2\sqrt{A}},$$

the convergence is rapid.

Cauchy Sequences

Definition

A sequence $\{p_n\}$ is said to be a *Cauchy Sequence* if, for any $\epsilon > 0$, there is a number N such that $\|p_n - p_m\| < \epsilon$ whenever both n and m are larger than N.

A Cauchy sequence is a sequence that looks as though it ought to converge. The main question about convergence is the existence of a point p to which p_n converges. The Monotone Convergence Property of the real numbers can be shown to imply that all Cauchy Sequences in $\mathbb{R}^n$ converge.

Example

Let

$$x_n = \int_1^n \frac{\cos t}{t^2}\,dt.$$

This is a Cauchy Sequence:

$$|x_n - x_m| = \left|\int_m^n \frac{\cos t}{t^2}\,dt\right| \leq \int_m^n \frac{1}{t^2}\,dt = \left|\frac{1}{m} - \frac{1}{n}\right|.$$

The Monotone Convergence Property also implies the Least Upper Bound property:

> *If S is a set of real numbers which is bounded above, then there is a real number x which is the least upper bound of S.*

Nested Closed Interval Theorem

The next theorem is very useful:

Theorem

Let $\{I_n\}$ be a sequence of non-empty bounded closed intervals in $\mathbb{R}$ with $I_{n+1} \subset I_n$. Then $\cap_{n=1}^{\infty} I_n \neq \phi$, i.e., there is at least one number $x \in \mathbb{R}$ such that $x \in I_n$ for every $n \in \mathbb{Z}^+$.

Proof.

Let $I_n = [a_n, b_n]$. Since $I_{n+1} \subset I_n$, $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$. Thus, the Monotone Convergence Property implies that there are real numbers a and b such that $a_n \to a$ and $b_n \to b$. We have $a = lub\,(a_n)$ and $b = glb\,(b_n)$. Since $b_n \geq a_m$ for all n, m, $b_n \geq a$, and finally $b \geq a$. Then $[a, b]$ is non-empty and a subset of every I_n. □

Bolzano-Weierstrass

Theorem (Bolzano-Weierstrass)

Every bounded infinite set of real numbers has a cluster point.

Proof.

Let S be an infinite subset of the interval $[a_1, b_1]$. Since $[a_1, b_1] = \left[a_1, \frac{a_1+b_1}{2}\right] \cup \left[\frac{a_1+b_1}{2}, b_1\right]$, at least one of these halves must contain infinitely many points of S. Define $a_2 < b_2$ so that $[a_2, b_2]$ is such a half. Continue in this fashion to construct a nested sequence of bounded closed intervals I_n, each of which contains infinitely many points of S, and with $b_n - a_n \to 0$. By the previous theorem, $\cap_{n=1}^{\infty} I_n$ is non empty with zero width, so it contains a single point, which must be a cluster point of S. □

Bounded sequence theorem

Theorem

Every bounded sequence of real numbers has a limit point, and therefore has a converging subsequence.

Cauchy Sequences Converge

Theorem

Any Cauchy sequence of real numbers is convergent.

Proof.

Let $\{x_n\}$ be a Cauchy sequence of real numbers. Then it must be bounded. Let $\alpha = \liminf x_n$ and let $\beta = \limsup x_n$. Let $\epsilon > 0$ be given. Since x_n is Cauchy, there is N such that for $n,\ m > N$, $|x_n - x_m| < \frac{\epsilon}{3}$. There are numbers $n_1,\ m_1 > N$ such that $|x_{n_1} - \alpha| < \frac{\epsilon}{3}$ and $|\beta - x_{m_1}| < \frac{\epsilon}{3}$. Then

$$|\beta - \alpha| \leq |\beta - x_{m_1}| + |x_{m_1} - x_{n_1}| + |x_{n_1} - \alpha| < \epsilon.$$

Since this is true for all $\epsilon > 0$, $\alpha = \beta$. As a consequence, $x_n \to \alpha$. □

Cauchy Sequences Converge in $\mathbb{R}^n$

Corollary

Any Cauchy sequence in $\mathbb{R}^n$ converges.

Proof.

If $p_k = (x_{1k}, \ldots, x_{nk})$, $k \in \mathbb{Z}^+$ is a Cauchy sequence in $\mathbb{R}^n$, then each x_{jk}, $k \in \mathbb{Z}^+$ is a Cauchy sequence in $\mathbb{R}$. Hence each $x_{jk} \to x_j$ Then $p_k \to (x_1, \ldots, x_n)$. (Exercise 35, Section 1.6). □

Many proofs that a problem has a solution proceed as follow:

1. Construct a sequence of approximate solutions.
2. Show that the constructed sequence lies in a *compact* subset of some space (which usually has infinite dimension).
3. Compactness guarantees that a subsequence must converge to a limit.
4. Show that the kind of convergence is good enough to ensure that the limit actually solves the problem.

Definition

A collection $\mathcal{S}$ of open sets $\mathcal{O}_\alpha$ is said to be an open covering of the set K if $K \subset \cup_\alpha O_\alpha$. The covering is said to be a finite covering if $\mathcal{S}$ consists of only a finite number of open sets.

Definition

A set K is called *compact* if every open covering of K can be reduced to a finite covering. This means that the must exist a finite sub-collection $\mathcal{S}_0$ of the original open sets which is still a covering of K.