Advanced Calculus

Professor David Wagner

[1]Department of Mathematics
University of Houston

September 4

Definitions

Recall from last lecture:

Definition

A collection $\mathcal{S}$ of open sets $\mathcal{O}_\alpha$ is said to be an open covering of the set K if $K \subset \cup_\alpha O_\alpha$. The covering is said to be a finite covering if $\mathcal{S}$ consists of only a finite number of open sets.

Definition

A set K is called *compact* if every open covering of K can be reduced to a finite covering. This means that the must exist a finite sub-collection $\mathcal{S}_0$ of the original open sets which is still a covering of K.

Theorem

The interval $[a, b]$ is compact.

- Suppose $\mathcal{S}$ is an open covering of $[a, b]$.
- If $x \in [a, b]$ then $\mathcal{S}$ also covers $[a, x]$.
- Let x be the $l.u.b.$ of $A = \{y : [a, y] \text{ has a finite subcover from } \mathcal{S}\}$.
- Since $a \in A$, A is not empty.
- Since $x \in [a, b]$, there is an open set $\mathcal{O} \in \mathcal{S}$ such that $x \in \mathcal{O}$.
- Since $\mathcal{O}$ is open, there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$.

- Since $x = l.u.b.(A)$, there is $y \in A \cap (x - \epsilon, x)$.
- Then there is a finite subcover from $\mathcal{S}$ of $[a, y]$: $\{U_1, \ \cdots, \ U_k\}$.
- Then $\{U_1, \ \cdots, \ U_k, \ \mathcal{O}\}$ is a finite subcover of $[a, x]$ from $\mathcal{S}$.
- Thus, $x \in A$.
- If $x < b$, then there is $z \in (x, b) \cap (x.x + \epsilon)$.
- Then $[a, z] \subset U_1 \cup \cdot \cup U_k \cup \mathcal{O}$.
- Thus, $x < b \implies x \neq l.u.b.(A)$. Since $x = l.u.b.(A)$, $x = b$.
- Thus $b \in A$, so $[a, b]$ has a finite subcover from $\mathcal{S}$.
- So $[a, b]$ is compact. □

Theorems about compact sets

Theorem (Heine-Borel)

The compact sets in $\mathbb{R}^n$ are exactly those that are closed and bounded.

Do exercises 1 and 2 on page 69 to show that every compact set in $\mathbb{R}^n$. To prove the converse:

Lemma (Spivak, *Calculus on Manifolds* p. 8)

If B is a compact subset of $\mathbb{R}^m$, $x \in \mathbb{R}^n$, and $\mathcal{O}$ is an open covering of $\{x\} \times B$, then there is an open set $U \subset \mathbb{R}^n$ such that $U \times B$ is covered by a finite number of sets in $\mathcal{O}$.

Proof of Lemma:

- Clearly $\{x\} \times B$ is compact.
- So there is a finite subcover of $\{x\} \times B$ from $\mathcal{O}$, which we will still label $\mathcal{O}$.
- Then we need only show that there is an open U such that $U \times B$ is covered by $\mathcal{O}$.
- For each $y \in B$ there is $W \in \mathcal{O}$ such that $(x, y) \in W$.
- Since W is open, there are open sets $U_y \subset \mathbb{R}^n$ and $V_y \subset \mathbb{R}^m$ such that $(x, y) \in U_y \times V_y \subset W$.
- The sets V_y cover B which is compact, so a finite number $V_{y1}, \cdots, V_{yk}$ also cover B.
- Let $U = U_{y1} \cap \cdots \cap U_{yk}$.
- If $(x', y') \in U \times B$, then $y' \in V_{yi}$ for some i, and $x' \in U_{yi}$.
- So $(x', y') \in U_{yi} \times V_{yi}$ which is a subset of some $W \in \mathcal{O}$.

Theorem

If A is a compact subset of $\mathbb{R}^n$ and B is a compact subset of $\mathbb{R}^m$, then $A \times B$ is a compact subset of $\mathbb{R}^{n+m}$.

Proof:

- If $\mathcal{O}$ is an open cover of $A \times B$, then for each $x \in A$, $\mathcal{O}$ covers $\{x\} \times B$.
- By the Lemma, there is an open set $U_x \subset \mathbb{R}^n$ such that $U_x \times B$ has an open subcover from $\mathcal{O}$.
- Since A is compact, A is covered by a finite set $\{U_{x1}, \cdots, U_{xk}\}$.
- Since U_{xi} is covered by a finite subset of $\mathcal{O}$, $A \times B$ is covered by a finite union of finite subsets of $\mathcal{O}$.
- Thus, $A \times B$ is compact.

Corollary

Any closed and bounded rectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ *is compact.*

Theorem

Any closed subset of a compact set is compact.

Corollary (Heine-Borel)

Every closed and bounded subset of $\mathbb{R}^n$ *is compact.*

More theorems about compact sets

Theorem (Bolzano-Weierstrass #1)

Any bounded infinite set in $\mathbb{R}^n$ has a cluster point.

- Let C be a bounded infinite subset of $\mathbb{R}^n$.
- Then $D = \overline{C}$ is closed and bounded, hence by Heine-Borel, D is compact.
- If C has no cluster point, then every point in D has an open neighborhood which contains only finitely many points of C.
- The collection of these open sets covers D, which is compact. So D is covered by a finite subset of this collection. But then C is a finite union of finite sets.
- Thus, ((C is bounded) $\implies$ ((C has no cluster point) $\implies$ (C is finite)).
- This is the contrapositive of (C is bounded) $\implies$ ((C is infinite) $\implies$ (C has a cluster point))

More theorems about compact sets

Theorem (Bolzano-Weierstrass #2)

Any bounded sequence in $\mathbb{R}^n$ has a limit point, and thus has a converging subsequence.

More theorems about compact sets

Theorem

If $C_1 \supset C_2 \supset C_3 \supset \cdots$ is a nested sequence of nonempty compact sets, then their intersection $\cap_{n=1}^{\infty} C_n$ is nonempty.

- Suppose $\cap_{n=1}^{\infty} C_n = \phi$. Then $\cup_{n=1}^{\infty} C_n^c \supset C_1$.
- Since C_1 is compact, and each C_n^c is open, a finite subcollection $C_{n_1}^C, \cdots, C_{n_k}^c$ covers C_1.
- Then $\cap_{j=1}^{k} C_{n_j} = C_{n_k} = \phi$.
- Thus C_n compact and $C_{n+1} \subset C_n \implies ((C_n \neq \phi) \implies (\cap_{n=1}^{\infty} C_n \neq \phi))$

Another theorem

Theorem

If S is any bounded subset of $\mathbb{R}^n$, and $\delta > 0$ is given, then it is possible to choose a finite set of points $p_1, \cdots, p_m$ in S so that any $p \in S$ is within distance δ of p_j for some j.

- S is covered by $\{B(p, \delta) : p \in S\}$.
- This collection also covers $\overline{S}$, which is compact.
- Then some finite subset $\{B(p_i, \delta) : i = 1, \cdots, m\}$ covers $\overline{S}$.
- Then every $p \in S$ is within distance δ of some p_i.