# Advanced Calculus

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# Sequential Continuity

## Definition

Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is said to be *sequentially continuous* at $p_0 \in D$ if, whenever $\{p_n\}$ is a sequence in $D$ with $\lim_{n\to\infty} p_n = p_0$, then $\lim_{n\to\infty} f(p_n) = f(p_0)$. We say that $f : D \to \mathbb{R}^m$ is sequentially continuous on $D$, if for every $p \in D$, $f$ is sequentially continuous at $p$.

## Theorem

*A function $f$ is continuous at $p \in D$ if and only if $f$ is sequentially continuous at $p$.*

# Sequential Continuity

Proof: $\implies$

- Suppose $f$ is continuous at $p$. Let $\{p_n\}$ be a sequence in $D$ with $\lim_{n\to\infty} p_n = p$.
- Let $\epsilon > 0$ be given. Since $f$ is continuous at $p$ there is $\delta > 0$ such that $x \in B(p, \delta) \cap D \implies \|f(x) - f(p)\| < \epsilon$.
- Since $\{p_n\} \to p$ there is $N \in \mathbb{R}$ such that $\|p_n - p\| < \delta$ for $n > N$. Also, $p_n \in D$.
- Then for $n > N$, $\|f(p_n) - p\| < \epsilon$.
- Thus, $f$ is sequentially continuous at $p$.

# Sequential Continuity

Proof: $\Longleftarrow$

- Suppose $f$ is not continuous at $p$. Then for some $\epsilon > 0$ and for each $n \in \mathbb{Z}^+$, there is $p_n \in D$ such that $\|p_n - p\| < \frac{1}{n}$ and $\|f(p_n) - f(p)\| \geq \epsilon$.
- Thus, $p_n \to p$ but $f(p_n) \not\to f(p)$.
- So, if $f$ is not continuous at $p$, then $f$ is not sequentially continuous at $p$.
- This is the contrapositive of the statement: *If $f$ is sequentially continuous at $p$, then $f$ is continuous at $p$.*

# Sequential Continuity

## Example

Let

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Then if $p_n = \left(\frac{1}{n}, \frac{c}{n}\right)$,

$$\lim_{n\to\infty} f(p_n) = f\left(\frac{1}{n}, \frac{c}{n}\right) = lim_{n\to\infty} \frac{\frac{c^2}{n^3}}{\frac{1}{n^2} + \frac{c^4}{n^4}} \quad (1)$$

$$= lim_{n\to\infty} \frac{\frac{c^2}{n}}{1 + \frac{c^4}{n^2}} = 0. \quad (2)$$

# Sequential Continuity

## Example (Continued)

But if $q_n = \left(\frac{1}{n^2}, \frac{1}{n}\right)$,

$$\lim_{n\to\infty} f\left(\frac{1}{n^2}, \frac{1}{n}\right) = lim_{n\to\infty}\frac{\frac{1}{n^4}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2}$$

So $f$ is not continuous at $(0,0)$.

# Relative Topology

**Definition**

Let $D$ be a subset of $\mathbb{R}^n$. We say that a subset $S$ of $D$ is *open relative to $D$* if there is an open subset $U$ of $\mathbb{R}^n$ such that $S = U \cap D$. $S$ is *closed relative to $D$* if there is a closed subset $C$ of $\mathbb{R}^n$ such that $S = C \cap D$. The collection of subsets of $D$ which are open relative to $D$ is called the *relative topology* or *subspace topology* on $D$.

# The algebra $C(D)$

## Theorem

*Let $C(D)$ denote the set of all continuous functions on $D$ with real values. Then $C(D)$ is closed under addition, scalar multiplication, and multiplication $(fg)(x) = f(x)g(x)$.*

## Remark

An *algebra* is a vector space that is closed under multiplication of vectors.

# Proof

- Let $f$ and $g$ be elements of $C(D)$. Then for any $p \in D$, and any sequence $p_n \to p$, we have $f(p_n) \to f(p)$ and $g(p_n) \to g(p)$.
- Then by standard results on real sequences, $f(p_n) + g(p_n) \to f(p) + g(p)$, $\alpha f(p_n) \to \alpha f(p)$, and $f(p_n) g(p_n) \to f(p) g(p)$.
- Thus, $f + g$, $\alpha f$, and $fg$ are in $C(D)$.

Similarly, one can show that if $f$ and $g$ are continuous on $D$, then $\frac{f}{g}$ is continuous at all $p \in D$ for which $g(p) \neq 0$.

- It is easy to show that constant functions and co-ordinate functions $x_1, \cdots, x_n$ on $\mathbb{R}^n$ are continuous.
- Then, by the previous theorem, all polynomial functions are continuous on $\mathbb{R}^n$, and
- all rational functions $\frac{p(x_1, \cdots, x_n)}{q(x_1, \cdots, x_n)}$ where $p$ and $q$ are polynomials, are continuous at all points $(x_1, \cdots, x_n)$ for which $q(x_1, \cdots, x_n) \neq 0$.

# Function Composition

## Theorem

*Let the function $g$ be continuous on a set $D \subset \mathbb{R}^n$ with values in $S \subset \mathbb{R}^m$. Let the function $f$ be continuous on $S$ with values in $\mathbb{R}^k$. Then the composite function $F$, given by*

$$F(p) = f(g(p))$$

*is continuous on $D$.*

# Proof

- Let $U$ be open in $\mathbb{R}^k$. Since $f$ is continuous on $S$, $f^{-1}(U)$ is open relative to $S$.
- Since $g$ is continuous on $D$ with values in $S$, $g^{-1}\left(f^{-1}(U)\right)$ is open relative to $D$.
- Thus, $F^{-1}(U) = g^{-1}\left(f^{-1}(U)\right)$ is relatively open in $D$.
- So $F = f \circ g$ is continuous on $D$.

# Examples

Let $A(x, y) = x + y$ and $M(x, y) = xy$. If we prove the continuity of $A$ and $F$, then for $f$ and $g$ in $C(D)$, then

1. The vector function $F(p) = (f, g)(p) = (f(p), g(p))$ is continuous on $D$,
2. Then $A \circ F(p) = f(p) + g(p)$ is continuous on $D$, and
3. $M \circ F(p) = f(p)g(p)$ is continuous on $D$.

# Trig functions

What, exactly, are the functions $\sin(x)$ and $\cos(x)$? We could define them as follows:

## Definition

The function $\cos(x)$ is the unique solution to the Initial Value Problem:

$$f''(x) + f(x) = 0, \quad f(0) = 1, \quad f'(0) = 0,$$

and the function $\sin(x)$ is the unique solution to:

$$f''(x) + f(x) = 0, \quad f(0) = 0, \quad f'(0) = 1,$$

# Intermediate Value Theorem

## Theorem

*Let $S$ be a connected set, and let $f : S \to \mathbb{R}^m$. Then $f(S)$ is connected.*

## Corollary (Intermediate Value Theorem)

*Let $f : [a, b] \to \mathbb{R}$ and suppose $f(a)f(b) < 0$. Then there is $c \in (a, b)$ such that $f(c) = 0$.*

## Corollary (IVT Version 2)

*Let $f : [a, b] \to \mathbb{R}$ and suppose $f(a) < d < f(b)$. Then there is $c \in (a, b)$ such that $f(c) = d$*

# Proof of Theorem

- We show that if $f$ is continuous on $S$, and $f(S)$ is not connected, then $S$ is not connected.
- If $f(S)$ is not connected, then $f(S) = A \cup B$, where $A$ and $B$ are mutually separated.
- This means that neither $A$ nor $B$ contains a boundary point of the other.
- Then both $A$ and $B$ are relatively open in $f(S)$–that is, there are open subsets $U$ and $V$ of $\mathbb{R}^m$ such that $A = U \cap f(S)$ and $B = V \cap f(S)$.
- Then $S = f^{-1}(A) \cup f^{-1}(B)$, and since $f$ is continuous on $S$, $f^{-1}(A)$ and $f^{-1}(B)$ are relatively open in $S$, hence they are mutually separated.
- Thus, if $f(S)$ is disconnected, then $S$ is disconnected. This is equivalent to the statement: *If $f$ is continuous on $S$ and $S$ is connected, then $f(S)$ is connected.*