Advanced Calculus

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Pathwise Connected

Definition

A set S is *pathwise connected* if every pair of points $p, \; q$ in S can be joined by a continuous path γ lying entirely in S.

Theorem

Any pathwise connected set S is connected.

Proof:

- Suppose S is pathwise connected, and $S = A \cup B$, with A and B non-empty.
- Choose $p \in A$ and $q \in B$, and join p and q by a path γ in S with $\gamma(0) = p$ and $\gamma(1) = q$.

- Let $A_0 = A \cap \gamma([0, 1])$, and $B_0 = B \cap \gamma([0, 1])$.
- If A and B are mutually separated, so are A_0 and B_0. In this case, $\gamma([0, 1])$ is disconnected.
- But $\gamma([0, 1])$ is connected, so A and B cannot be mutually separated.
- Thus, S is connected.

Theorem

No continuous function can map the open unit square S 1-1 and onto the interval $[0,1]$.

Proof:

- Suppose ϕ maps S continuously onto $[0,1]$. Let $p \neq q$ be points in $[0,1]$.
- Then there are points $a,\ b \in S$ such that $\phi(a) = p$ and $\phi(b) = q$.
- Let α and β be continuous paths in S such that $\alpha(0) = \beta(0) = a$ and $\alpha(1) = \beta(1) = b$.
- Let c be any number between p and q. Then there must be at least 2 points $r,\ s \in S$ such that r is on the trace of α, s is on the trace of β, and $\phi(r) = \phi(s) = c$.
- Thus, ϕ cannot be 1-1.

Theorem

Let $f : I = [a, b] \to \mathbb{R}$ be continuous and 1-1. Then f is strictly monotonic on I.

Proof:

- We show that if f is not strictly monotonic, then f is not 1-1.
- Suppose $f(a) < f(b)$. Take any x with $a < x < b$.
- If $f(x) > f(b)$ then by the IVT there must be c, $a < c < x$, such that $f(x) = f(b)$.
- If $f(x) = f(b)$ or $f(x) = f(a)$, we are done.
- if $f(x) < f(a)$ there must be c, $a < c < x$, such that $f(x) = f(a)$.
- Thus, if f is 1-1, then $f(a) < f(x) < f(b)$.
- Now apply this argument with any $x_1 < x_2$ in I to see that f must be strictly increasing on I.

Uniform Continuity

Definition

We say that a function f is *uniformly continuous* on a set E if and only if, for each $\epsilon > 0$ there is $\delta > 0$ such that $|f(p) - f(q)| < \epsilon$ whenever p and q are in E, and $|p - q| < \delta$

Example

- $f(x) = 5x$ is uniformly continuous on $\mathbb{R}$. In this case, $\delta = \frac{\epsilon}{5}$.
- If $p > 0$, $g(x) = \frac{1}{x}$ is uniformly continuous on $[1, \infty)$, with $\delta = \frac{\epsilon}{p^2}$.
- $g(x) = \frac{1}{x}$ is continuous but not uniformly continuous on $(0, 1)$.
- $h(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Theorem

If E is a compact set and f is continuous on E, then f is uniformly continuous on E.

Proof:

- Suppose f is not uniformly continuous.
- Then there is $\epsilon > 0$ such that for any $\delta > 0$ there are points $p, \ q \in E$ such that $\|p - q\| < \delta$ and $|f(p) - f(q)| \geq \epsilon$.
- Choose $p_n, \ q_n$ which correspond to $\delta = \frac{1}{n}$.
- Since E is compact, by Bolzano-Weierstrass there are convergent subsequences $p_n \to p \in E$ and $q_n \to q \in E$.
- Then $\|p - q\| \leq \|p - p_n\| + \|p_n - q_n\| + \|q_n - q\|$ which converges to 0 as $n \to \infty$. Thus $p = q$ but $\|f(p_n) - f(q_n)\| \geq \epsilon$.
- Thus, if f is not uniformly continuous on E, then f is not continuous on E.

Uniform Convergence

Definition

Let $\{f_n\}$ be a sequence of functions which map $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. We say that f_n converges *uniformly on D* to a function $f : D \to \mathbb{R}^m$, if

$$\sup_{x \in D} \|f_n(x) - f(x)\| \to 0 \text{ as } n \to \infty.$$

Examples

Let $A(x, y) = x + y$ and $M(x, y) = xy$. If we prove the continuity of A and F, then for f and g in $C(D)$, then

1. The vector function $F(p) = (f, g)(p) = (f(p), g(p))$ is continuous on D,
2. Then $A \circ F(p) = f(p) + g(p)$ is continuous on D, and
3. $M \circ F(p) = f(p)g(p)$ is continuous on D.

Trig functions

What, exactly, are the functions $\sin(x)$ and $\cos(x)$? We could define them as follows:

Definition

The function $\cos(x)$ is the unique solution to the Initial Value Problem:

$$f''(x) + f(x) = 0, \quad f(0) = 1, \quad f'(0) = 0,$$

and the function $\sin(x)$ is the unique solution to:

$$f''(x) + f(x) = 0, \quad f(0) = 0, \quad f'(0) = 1,$$

Intermediate Value Theorem

Theorem

Let S be a connected set, and let $f : S \to \mathbb{R}^m$. Then $f(S)$ is connected.

Corollary (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ and suppose $f(a)f(b) < 0$. Then there is $c \in (a, b)$ such that $f(c) = 0$.

Corollary (IVT Version 2)

Let $f : [a, b] \to \mathbb{R}$ and suppose $f(a) < d < f(b)$. Then there is $c \in (a, b)$ such that $f(c) = d$

Proof of Theorem

- We show that if f is continuous on S, and $f(S)$ is not connected, then S is not connected.
- If $f(S)$ is not connected, then $f(S) = A \cup B$, where A and B are mutually separated.
- This means that neither A nor B contains a boundary point of the other.
- Then both A and B are relatively open in $f(S)$–that is, there are open subsets U and V of $\mathbb{R}^m$ such that $A = U \cap f(S)$ and $B = V \cap f(S)$.
- Then $S = f^{-1}(A) \cup f^{-1}(B)$, and since f is continuous on S, $f^{-1}(A)$ and $f^{-1}(B)$ are relatively open in S, hence they are mutually separated.
- Thus, if $f(S)$ is disconnected, then S is disconnected. This is equivalent to the statement: *If f is continuous on S and S is connected, then $f(S)$ is connected.*