Advanced Calculus

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Definition

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Suppose $f : D \subset \mathbb{R}^n \to \mathbb{R}^k$ and suppose $a \in \overline{D}$. We say that

$$\lim_{x \to a} f(x) = L,$$

if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\|f(x) - L\| < \epsilon \text{ whenever } 0 < \|x - a\| < \delta.$$

Remark

Note that the value of $f(a)$ is not relevant to the existence or value of the limit. $f(A)$ does not even need to be defined for the limit to exist.

Theorem

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Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^k$ and suppose $a \in \overline{D}$. Then $\lim_{x \to a} f(x) = L$, if an only if, for each sequence $p_n \in D$ such that $p_n \to a$ and $p_n \neq a \; \forall n$, for every $\epsilon > 0$ there is N such that $\|f(p_n) - L\| < \epsilon$ whenever $n > N$.

Proof: Exercise!

Examples

Example

Let

$$f(x,y) = \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \text{ and let } f(0,0) = 0.$$

Then $lim_{(x,y)\to(0,0)} f(x,y)$ does not exist: If $p_n = (0, \frac{1}{n})$, $p_n \to (0,0)$ and $f(p_n) = 0$. If $q_n = \left(\frac{1}{n}, \frac{1}{n}\right)$, then $q_n \to (0,0)$ and $f(q_n) = \frac{1}{2}$.

Examples

Example

Let

$$g(x, y) = \frac{x^2 y}{x^4 + y^2}, \text{ if } (x, y) \neq (0, 0), \text{ and let } f(0, 0) = 0.$$

As in the previous example, we have $f(p_n) = 0$, but now $f(q_n) = \frac{\frac{1}{n^3}}{\frac{1}{n^4} + \frac{1}{n^2}} \to 0$. But f still has no limit as $(x, y) \to (0, 0)$, because if $r_n = \left(\frac{1}{n}, \frac{1}{n^2}\right)$, $r_n \to (0, 0)$ but $f(r_n) = \frac{1}{2}$

Examples

Example

Let

$$h(x,y) = \frac{x\left(x^2 - y^2\right)}{x^2 + y^2}, \text{ if } (x,y) \neq (0,0)\,, \text{ and let } f(0,0) = 0.$$

Now

$$|h(x,y) - 0| \leq \frac{|x|\left(x^2 + y^2\right)}{x^2 + y^2} = |x| \to 0 \text{ as } (x,y) \to (0,0).$$

Definition

Recall from Calculus I:

Definition

Let $f : D \subset \mathbb{R} \to \mathbb{R}$.If a is interior to D, we say that f is differentiable at a if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists. In this case we say that the limit is the derivative of f at a, denoted $f'(a)$.

Remark

If $f'(a)$ exists, then $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} - f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)-f'(a)h}{h} = 0.$

If we replace $f'(a)$ with any other number M, we get

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} - M = \lim_{h\to 0} \frac{f(a+h)-f(a)-Mh}{h} = \left(f'(a)-M\right).$$

This means that $f'(a)h$ is the best linear approximation to $f(a+h)-f(a)$, which in most uses is a nonlinear function. Another way to explain this is to examine:

$$f(a+h)-f(a)-f'(a)h = \left(\frac{f(a+h)-f(a)}{h} - f'(a)\right) h = \epsilon(h)h$$

where $\epsilon(h) = f(a+h)-f(a)-f'(a)h \to 0$ as $h \to 0$

Again, if we replace $f'(a)$ with M, we don't get $\epsilon(h)h$. We get $(f'(a) - M)\, h$, and $f'(a) - M$ does not tend to 0 as $h \to 0$.

Example, Theorem

Let $f(x) = x^{\frac{3}{2}}$. Then $f(x) - f(0) - 0x = \sqrt{x}x$. Here $\epsilon(x) = \sqrt{x} \to 0$ as $x \to 0+$. So f has a (right side) derivative, 0, at 0.

Theorem

If $f : D \subset \mathbb{R} \to \mathbb{R}$ and a is an interior point of D, and if $f'(a)$ exists, then f is continuous at a.

Proof.

$$f(x) - f(a) = f'(a)(x-a) + \epsilon(x-a)(x-a) \to 0 \text{ as } x \to a.$$

□

Product Rule

Theorem

Suppose f and g are differentiable at a. Then so is $p(x) = f(x)g(x)$, and $p'(a) = f'(a)g(a) + f(a)g'(a)$.

Proof

- $p(a+h) - p(a) = f(a+h)g(a+h) - f(a)g(a) = f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)$
- $= (f(a+h) - f(a))\, g(a+h) + f(a)\,(g(a+h) - g(a)) = (f'(a)h + \epsilon_1(h)h)\, g(a+h) + f(a)\,(g'(a)h + \epsilon_2(h)h)$.
- $= (f'(a)g(a+h) + f(a)g'(a))\, h + (\epsilon_1(h)g(a+h) + \epsilon_2(h)f(a))\, h$.
- Thus, the first order term is $(f'(a)g(a+h) + f(a)g'(a))\, h$ and the rest is $\epsilon(h)h$ since g is continuous at a.

Min/Max points

Theorem

Suppose f is defined on an open interval I and $a \in I$. If f has a local maximum or minimum at a and f is differentiable at a, then $f''(a) = 0$.

Proof.

Suppose f has a local minimum at a. Then for some $\epsilon > 0$, $f(x) - f(a) \geq 0$ for $a \in (a - \epsilon, a + \epsilon)$. Thus, for such x, $\frac{f(x)-f(a)}{x-a}$ has the same sign as $x - a$. Thus, $f'(a)$ is a limit of non-negative numbers on the right and of non-positive numbers on the right. Thus, if $f'(a)$ exists, it must be 0. □

Rolle's Theorem

Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, there is at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

By the Extreme Value Theorem, f must attain a minimum value and a maximum value on the compact set $[a, b]$. If both are attained at the endpoints, then the maximum equals the minimum, so f is constant on $[a, b]$, and $f'(x) = 0$ for $x \in (a, b)$. Otherwise, an extreme value is attained at an interior point c, so that $f'(c) = 0$ by the previous theorem. □

Mean Value Theorem

Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof.

The line through $(a, f(a))$ and $(b, f(b))$ has slope $\frac{f(b)-f(a)}{b-a}$. Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a, b]$ and differentiable on (a, b). $g(b) = g(a)$, so Rolle's Theorem states that there must be $c \in (a, b)$ such that $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. □

Applications of MVT

Definition

We say that a function f is *increasing* (respectively *strictly increasing*) on an interval I, if $f(a) \leq f(b)$ (resp. $f(a) < f(b)$ whenever $a,\ b \in I$ and $a < b$. We define *decreasing* and *strictly decreasing* functions similarly.

Theorem (Interpretation of the derivative)

Suppose f is differentiable on the open interval I.

a *If $|f'(x)| \leq C$ for all $x \in I$, then $|f(b) - f(a)| \leq C\,|b - a|$ for all $a,\ b \in I$.*

b *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

c *If $f'(x) \geq 0$ (resp. $f'(x) > 0$, $f'(x) \leq 0$, or $f'(x) < 0$) for all $x \in I$, then f is increasing (resp. strictly increasing, decreasing, strictly decreasing) on I.*

Proof

Proof.

Let $a,\ b \in I$. Since f is differentiable on I and $[a, b] \subset I$, f is continuous on I. Then the MVT gives us a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. For a, $|f'(c)| \leq C$, so $|f(b) - f(a)| \leq C\,|b - a|$. Parts b, c are proved similarly. □

Remark

If all we know about f is that f is differentiable at a and $f'(a) > 0$, it does *not* follow that f is increasing in some neighborhood of a.

Generalized MVT

Theorem (Generalized MVT)

Suppose that f and g are continuous on $[a, b]$, differentiable on (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof of Generalized MVT

Proof

- Let $h(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a))$.
- Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$.
- By Rolle's Theorem there is $c \in (a, b)$ such that $h'(c) = (f(b) - f(a))\,g'(c) - (g(b) - g(a))\,f'(c) = 0$.
- Since g' is never 0 on (a, b), $g'(c) \neq 0$ and $g(b) - g(a) \neq 0$ (by MVT).
- Then dividing by $g'(c)\,(g(b) - g(a))$ gives the result.

Remark

If we use f and g to parameterize a curve: $y = f(t)$, $x = g(t)$, $t \in [a, b]$, then by the chain rule, $\frac{dy}{dx} = \frac{f'(t)}{g'(t)}$. In this case the Generalized MVT says that there is a tangent line to the parameterized curve, that is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$.

Application of Generalized MVT

Theorem (L'Hôpital's Rule I)

Suppose f and g are differentiable functions on (a, b) and

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0.$$

If g' never vanishes on (a, b) and

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L,$$

then g never vanishes on (a, b) and

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$