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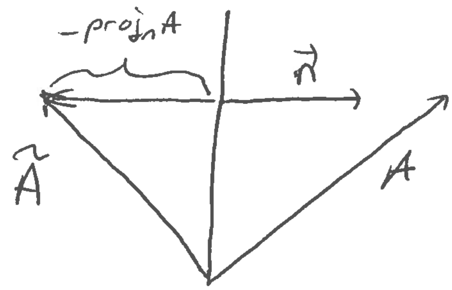
Math 3335 Homework Solutions

p 34 #14 The vector $\vec{n} = (3\vec{i} + 2\vec{j} + 6\vec{k})\frac{1}{7}$ is perpendicular to a plane. A line segment representing the vector $\vec{A} = 2i + 5j + 6k$ lies on one side of this plane. Regarding the plane as a mirror, write down the vector represented by the mirror image of $\vec{A}$.

Solution



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$\tilde{A}$ = mirror image of A

$$= \vec{A} - \underline{\underline{2}}\, \text{proj}_n \vec{A} = \vec{A} - \underline{\underline{2}}\, \frac{(A \cdot n)}{|n|^2}\vec{n}.$$

Since $|n| = 1$, $\tilde{A} = 2i + 5j + 6k - 2\,\frac{(6 + 10 + 36)}{7} \cdot \vec{n}$

$$= 2i + 5j + 6k - \frac{2}{7} \cdot 42 \cdot (3\vec{i} + 2\vec{j} + 6\vec{k})\frac{1}{7}$$

$$= 2i + 5j + 6k - \frac{36}{7}i - \frac{24}{7}j - \frac{72}{7}k$$

$$= -\frac{22}{7}i + \frac{11}{7}j - \frac{30}{7}k.$$

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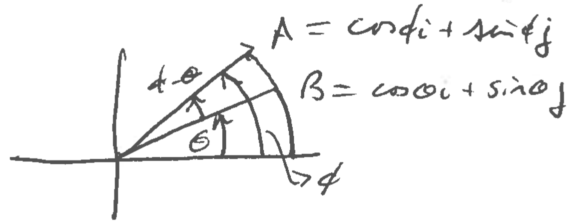
p34 #18 Let $\vec{A} = \cos\phi\,\vec{i} + \sin\phi\,\vec{j}$ and $\vec{B} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$. Draw these vectors in the xy plane. By interpreting the scalar product $A \cdot B$ geometrically, prove that $\cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$.

Proof



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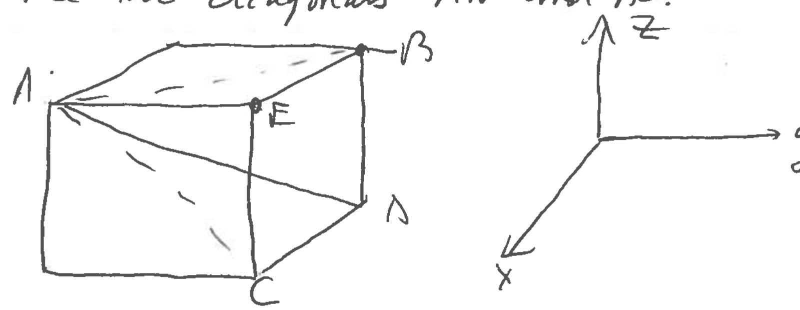
$$A \cdot B = \|A\|\,\|B\| \cos(\phi-\theta) = \cos(\phi-\theta)$$
$$= \cos\phi\cos\theta + \sin\phi\sin\theta \quad (\text{compute } A\cdot B).$$
So $\cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$.

p34 #23 Consider the cube in fig 1.27. Find the angles between

a The face diagonals $\vec{AB}$ and $\vec{AC}$.

8 Assume edge length is 1.



$$\vec{AB} = (-1, 1, 0) = -i + j$$
$$\vec{AC} = (0, 1, -1) = j - k$$
$$\cos(\text{angle}) = \frac{\vec{AB}\cdot\vec{AC}}{\|AB\|\,\|AC\|} = \frac{1}{2}$$

So angle $= \pi/3$ rad or $60°$.

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p34 23b The angle between

b the principal diagonal $\vec{AD}$ and the face diagonal $\vec{AB}$.

Solution

$AD = (-1, 1, -1) = -i + j - k$.

$AB = -i + j$

(8) $\cos(\text{angle}) = \dfrac{AD \cdot AB}{\|AD\| \, \|AB\|} = \dfrac{2}{\sqrt{3} \cdot \sqrt{2}} = \sqrt{\frac{2}{3}}$

$= \dfrac{\sqrt{6}}{3}$.

So angle $= \cos^{-1}(\sqrt{6}/3) = \cos^{-1}(.8165)$

$= 35.26^\circ$ or $.6155$ radians.

c. The principal diagonal AD and the edge AE

Solution

$\cos(\text{angle}) = \dfrac{AD \cdot AE}{\|AD\| \, \|AE\|}$ $\quad AE = (0, 0, 1)$.

(8) $= \dfrac{(-i + j - k) \cdot k}{\sqrt{3} \cdot 1} = \dfrac{1}{\sqrt{3}}$

angle $= \cos^{-1}(1/\sqrt{3}) = \cos^{-1}(.57735)$

$= 54.7356^\circ$ or $.9553$ radians.

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p34 #25. Prove: the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

Solution Suppose the rectangle is a square.
(8)
Then it is a parallelogram spanned by two perpendicular vectors $\vec{A}$ and $\vec{B}$ of equal length.
The diagonals are $\overrightarrow{A+B}$ and $\overrightarrow{A-B}$, and
$(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = |A|^2 - |B|^2 - A\cdot B + B\cdot A$
so the diagonals are perpendicular.
Similarly if $(A+B)\cdot(A-B) = |A|^2 - |B|^2 = 0$ then $|A| = |B|$ and the rectangle is a square.

p34 #28 By vector methods, show that the angle subtended at the circumference by a diameter of a circle is a right angle.

Solution

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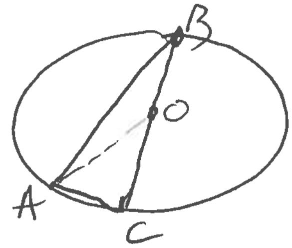

$\vec{BC}$ is diameter

Show $AB\cdot AC = 0$.

Let Center of circle $= \vec{0}$.

Then $AB = B - A$ $\quad B = -C$.
$AC = C - A$

$AB\cdot AC = (-C-A)(C-A) = |A|^2 - |C|^2 = r^2 - r^2 = 0$

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HW#2 problem #1 Express $2i - j + 3k$ as a sum of a vector parallel, plus a vector perpendicular, to $2i + 4j - 2k$.

Solution.

$\vec{A} = 2i - j + 3k$; $B = 2i + 4j - 2k$

(8)

$$A = \underbrace{\text{proj}_B A}_{\text{parallel}} + \underbrace{(A - \text{proj}_B A)}_{\perp}$$

$$\text{proj}_B A = \frac{(A \cdot B)}{\|B\|^2}\vec{B} = \frac{(4-4-6)}{4+16+4}(2i + 4j - 2k)$$

$$= \left(\frac{-6}{24}\right)(2i + 4j - 2k) = \left(-\tfrac{1}{2}i - j + \tfrac{1}{2}k\right) = A^{\parallel}$$

Then $A^{\perp} = A - A^{\parallel} = 2\vec{i} - \vec{j} + 3\vec{k} - \left(-\tfrac{1}{2}\vec{i} - \vec{j} + \tfrac{1}{2}\vec{k}\right)$

$$= \frac{5}{2}\vec{i} + \frac{5}{2}\vec{k}$$

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HW 2 # 2. Find the dihedral angle between the planes

$$2x + y - 2z = 5$$
$$3x - 4y = 2.$$

8 Solution $n_1 = 2i + j - 2k$ $\quad \cos\theta = \frac{n_1 \cdot n_2}{\|n_1\| \|n_2\|}$

$n_2 = 3i - 4j$

$$= \frac{6-4}{3 \cdot 5} = \frac{2}{15}$$

$\theta = \cos^{-1}(2/15) = 82.34^\circ$ or 1.437 radians.

3. Find a vector parametric equation for the line of intersection of the two planes in #2

Solution A direction vector can be found as $d = n_1 \times n_2$

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$$= \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 3 & -4 & 0 \end{vmatrix}$$

$= i(-8) - j(6) + k(-8-3) = -8i - 6j - 11k$

or $8i + 6j + 11k$.

To find a point in the intersection, find one solution to

$$2x + y - 2z = 5$$
$$3x - 4y \quad = 2$$

Note z is "eliminated" from 2^{nd} eqtn. Treat x as a free variable, y and z as dependent variables.

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HW 2 # 3, cont. Since x is free, choose $x=0$,

Then $-4y=2$, $y=-1/2$.

$-2z=5-y=11/2$.

$z=-\frac{11}{4}$.

$(x,y,z)=(0,-1/2,-11/4)$. (other solutions possible).

With $d=8i+6j+11k$, we have

$(x,y,z)=(0,-1/2,-11/4)+t(8,6,11)$.

or $-1/2 j-11/4 k+t(8i+6j+11k)$

4 Find the distance between planes

$$2x+y-2z=5$$

$$2x+y-2z=-1.$$

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Solution. Both planes are parallel, with normal vector $2i+j-2k, =n$

Then for (x_1,y_1,z_1) in plane #1, $n\cdot(x_1,y_1,z_1)=5$

and for (x_2,y_2,z_2) in plane #2,

$$n\cdot(x_2,y_2,z_2)=-1.$$

$$\text{dist}=\frac{|5-(-1)|}{|n|}=\frac{6}{3}=2.$$

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HW2 #5 Find the distance from the point $(1,-2,3)$ to the ~~line~~ plane $\frac{x}{2}+y-z=0$.

Solution. The plane has normal vector

$$\tfrac{1}{2}i+j-k=n.$$

$n\cdot(x,y,z)=0$ for pts on the plane.

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$$\text{So dist} = \frac{n\cdot(1,-2,3)}{\|n\|} = \frac{|1/2-2-3|}{\sqrt{\frac{1}{4}+1+1}} = \frac{4.5}{(3/2)}$$

3

#6 Find an equation for the plane through the points $\underset{P}{(1,0,-1)}$, $\underset{Q}{(2,0,1)}$, $\underset{R}{(1,1,0)}$.

Solution $A=\vec{PQ}=(1,0,2)$

$B=\vec{PR}=(0,1,1)$.

8

$$\text{Normal } N = A\times B = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= i(-2)-j(1)+k\cdot 1 \quad = -2i-j+k.$$

Then an equation is $N\cdot((x,y,z)-P)=0$

or $-2(x-1)-y+(z+1)=0$.

or $-2x-y+z+3=0$.

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HW2 # 7 Find the altitude of a parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$, if the base is the parallelogram determined by $\vec{a}$ and $\vec{b}$, and if

$$\vec{a} = (1,0,1)$$
$$\vec{b} = (0,2,1)$$
$$\vec{c} = (1,3,0)$$

Solution The altitude is $\dfrac{|c\cdot(a\times b)|}{\|a\times b\|}$

8

$$= \frac{\left|\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix}\right|}{\left\|\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}\right\|} = \frac{|-3-2|}{\| i(-2) - j\cdot 1 + k\cdot 2\|}$$

$$= \frac{5}{3}$$