Math 3335 Homework Solutions

p95 #10 A point moves along a curve so that its position R at time t is given by:

$$R(t) = t^2 i + t^2 j + t k.$$

Find

a. its speed v

$$= \|R'(t)\| \quad R'(t) = 2t i + 2t j + k$$

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$$\|R'(t)\| = \sqrt{4t^2 + 4t^2 + 1}$$
$$= \sqrt{8t^2+1}$$

b. the unit tangent T to its path.

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$$T(t) = \frac{R'(t)}{\|R'(t)\|} = \frac{2t i + 2t j + k}{\sqrt{8t^2+1}}$$

c. The vector $\kappa \vec{N}$.

$$\kappa \vec{N} = dT/ds = \frac{dT/dt}{ds/dt}.$$

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$$\frac{dT}{dt} = \frac{2i + 2j}{\sqrt{8t^2+1}} + \frac{2t i + 2t j + k}{(8t^2+1)^{3/2}} \cdot \left(-\tfrac{1}{2}\right) \cdot (16t)$$

$$= \frac{(2i+2j)(8t^2+1) - (2t i + 2t j + k)(8t)}{(8t^2+1)^{3/2}}$$

$$= \frac{2i + 2j - 8kt}{(8t^2+1)^{3/2}} \quad \Big| \quad \frac{dT}{ds} = \frac{2i+2j-8tk}{(8t^2+1)^2}$$

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p 95 #14 The position vector of a particle is given by

$$R(t) = \sqrt{2}\cos(3t)i + \sqrt{2}\cos(3t)j + 2\sin(3t)k.$$

Find its speed, the curvature and torsion of its path, and describe the path geometrically.

Solution $R'(t) = -3\sqrt{2}\sin(3t)i - 3\sqrt{2}\sin(3t)j + 6\cos(3t)k$

(4) $\|R'(t)\| = \sqrt{18\sin^2 3t + 18\sin^2(3t) + 36\cos^2(3t)}$
$= 6. =$ speed.

(4) $T(t) = \dfrac{R'(t)}{\|R'(t)\|} = -\frac{\sqrt{2}}{2}\sin(3t)i - \frac{\sqrt{2}}{2}\sin(3t)j + \cos(3t)k.$

(4) $k = \left\|\frac{dT}{ds}\right\| = \dfrac{\left\|\frac{dT}{dt}\right\|}{ds/dt} = \frac{1}{6}\left\|-\frac{3\sqrt{2}}{2}\cos(3t)i - \frac{3\sqrt{2}}{2}\cos(3t)j - 3\sin(3t)k\right\|$
$= \frac{1}{6}\sqrt{\frac{9}{2}\cos^2(3t) + \frac{9}{2}\cos^2(3t) + 9\sin^2(3t)} = \frac{3}{6} = \frac{1}{2}.$

(4) $N = \dfrac{dT}{ds}\dfrac{1}{k} = 2\,\dfrac{dT/dt}{ds/dt} = \frac{1}{3}\left(-\frac{3\sqrt{2}}{2}\cos 3t\, i - 3\frac{\sqrt{2}}{2}\cos(3t)j - 3\sin(3t)k\right)$
$= -\frac{\sqrt{2}}{2}\cos 3t\, i - \frac{\sqrt{2}}{2}\cos(3t)j - \sin(3t)k.$

$\frac{dN}{ds} = -kT + \tau B$, and N, T, B are orthonormal

(4) So $\tau = B \cdot \frac{dN}{ds}$. $B = T \times N = \begin{vmatrix} i & j & k \\ -\frac{\sqrt{2}}{2}\sin 3t & -\frac{\sqrt{2}}{2}\sin 3t & \cos 3t \\ -\frac{\sqrt{2}}{2}\cos 3t & -\frac{\sqrt{2}}{2}\cos 3t & -\sin 3t \end{vmatrix}$

$B = i\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} + 0k$

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p 95 #14, continued Then

4) $\frac{dN}{ds} = \frac{\frac{dN}{dt}}{\frac{ds}{dt}} = \frac{1}{6}\left(\frac{3\sqrt{2}}{2}\sin 3t\, i + 3\frac{\sqrt{2}}{2}\sin 3t\, j - 3\cos(3t)k\right)$

$= \frac{\sqrt{2}}{4}\sin 3t\, i + \frac{\sqrt{2}}{4}\sin 3t\, j - \frac{1}{2}\cos(3t)k$

4) $\frac{dN}{ds}\cdot B = \frac{dN}{ds}\cdot\left(i\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}\right) = \frac{2}{8}\sin 3t - \frac{2}{8}\sin(3t) = 0.$

(Alt: $\frac{dB}{ds} = 0 = -\tau N$). So $\tau = 0$.

4) The curve is a circle in the plane $\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = 0$ or $x - y = 0$. (Note that $B \cdot R(t) = 0$).

The center is $(0,0,0)$ and the radius is $\|R(t)\| = 2$.

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p 95 #16 If C is the curve given parametrically by

$$R(t) = \cos t\, i + \sin t\, j + 2t\, k$$

find

a. the normal N and the binormal B for this curve at $t=0$.

Solution $R'(t) = -\sin t\, i + \cos t\, j + 2K$

(4) $\|R'(t)\| = \sqrt{5}$

$T(t) = \frac{-\sin t}{\sqrt{5}} i + \frac{\cos t}{\sqrt{5}} j + \frac{2}{\sqrt{5}} k$. $T(0) = \frac{j+2k}{\sqrt{5}}$

(4) $T'(t) = -\frac{\cos t}{\sqrt{5}} i - \frac{\sin t}{\sqrt{5}} j + 0$

$N = \frac{T'(t)}{\|T'(t)\|} = -\cos t\, i - \sin t\, j$

(unit vector)

$N(0) = -i$

(4) $B(0) = T(0) \times N(0) = \left(\frac{1}{\sqrt{5}} j + \frac{2}{\sqrt{5}} k\right) \times (-i)$

$= \frac{1}{\sqrt{5}} (-j \times i - k \times i) = \frac{1}{\sqrt{5}} (k - 2j)$

$= \frac{1}{\sqrt{5}} (-2j + k)$

b. The equation of the plane through the point $R(0)$ and parallel to both vectors N and B of part (a).

(6) Solution This plane is normal to $T(0) = \frac{j+2k}{\sqrt{5}}$

$R(0) = i$ or $(1,0,0)$

The eqn is $T(0) \cdot (x-1, y, z) = 0$

or $\frac{1}{\sqrt{5}} (y + 2z) = 0$.