Math 3335 Homework Solutions

p112 #8 A volcano just erupted and lava is streaming down from the mountain top. Suppose that the altitude of the mountain is given by $z(x,y) = h e^{-(x^2+2y^2)}$ where h is the maximum height, and suppose also that the lava flows in the direction of steepest descent (fastest change in z). Find

a. the projection on the xy plane of the direction in which the lava flows away from the point $(1, 2, he^{-9})$.

Solution. The flow obeys

$$\text{①} \quad \frac{dx}{dt} = -\frac{\partial z}{\partial x} = -2x\,z$$

$$\text{②} \quad \frac{dy}{dt} = -\frac{\partial z}{\partial y} = -4y\,z$$

$$\text{③} \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

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① + ② $\Rightarrow$ $dy/dx = 2\,y/x$,

or $\quad dy/dx = (dy/dt)/(dx/dt)$

$$\frac{dy}{dx} = 2\cdot 2/1 \quad \text{at } (x,y) = (1,2).$$

So the flow has direction $\dfrac{\vec{i} + 4\vec{j}}{\sqrt{17}}$

b. the equation of the projection on the xy plane of the flow line of the lava passing through the point $(1, 2, he^{-9})$.

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p112 8b Solution $\frac{dy}{dx} = 2y/x$

(8) $\frac{dy}{y} = 2\frac{dx}{x} \to \int \frac{dy}{y} = 2\int \frac{dx}{x} \to \ln y = 2\ln x + c$

$y = x^2 e^c$ $\quad y = 2$ when $x = 1$

$2 = e^c$

$\underline{y = 2x^2}$.

p112 #22 If $\phi(x,y,z) = x^2y + zy + z^3$, find

a. The gradient of ϕ

(4) Solution $\nabla\phi = 2xy\,i + (x^2+z)\,j + (y+3z^2)\,k$.

b. The equation of the plane passing through the point $(1,-1,1)$ and tangent to the level surface of ϕ at that point.

Solution $\nabla\phi(1,-1,1) \perp$ level surface

(8) $\nabla\phi(1,-1,1) = -2i + 2j + 2k$.

So

$-2(x-1) + 2(y+1) + 2(z-1) = 0$ or

$-(x-1) + (y+1) + (z-1) = 0$ or

$-x + y + z + 1 = 0$.

Math 3335 Homework Solutions

p112 #31 Find the point on the sphere $x^2+y^2+z^2=84$ that is nearest the plane $x+2y+4z=77$.

Solution the normal vector to the plane is $i+2j+4k$.

At the point on the sphere closest to the plane, the normal vector to the sphere must be normal to the plane (i.e. parallel to $i+2j+4k$.

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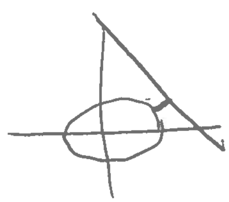

Since the radius is normal to the sphere, we seek

$(x,y,z)=(x,2x,4x)$ on the sphere

So we solve $x^2+4x^2+16x^2=84$

$21x^2=84 \quad x^2=4,$

$x=2,\ y=4,\ z=8.$

Note for this point $x+2y+4z=2+8+32=42<77$

$\Leftrightarrow$ the plane does not intersect the sphere.

Math 3335 Homework Solutions

p 117 # 2. Let $F = x^2 i + y^2 j + k$

a. Find the general equation of a flow line.

Solution Solve

$$\frac{dx}{dt} = x^2$$

$$\frac{dy}{dt} = y^2$$

(10)

$$\frac{dz}{dt} = 1$$

$\frac{dx}{x^2} = dt = \frac{dy}{y^2}$ so $\frac{-1}{x} = -\frac{1}{y} + c. = \frac{cy-1}{y}$

$x = \frac{-y}{cy-1} = \frac{y}{1-cy}.$

$\frac{dz}{dt} = 1$ so $z = z_0 + t.$

$\frac{dx}{x^2} = dt \Rightarrow \frac{-1}{x} = t + c_2$

$x = \frac{-1}{t+c_2}$, $y = \frac{-1}{t+c_3}$

$x = \frac{-1}{z-z_0+c_2}$ $y = \frac{-1}{z-z_0+c_3}$

(b.) Find the flow line through the point (1,1,2)

(6)

$x = 1$ when $y = 1$ so

$1 = \frac{1}{1-c\cdot 1} \Rightarrow c = 0.$

$x = y.$

$x = 1$ when $z = 2$, $t = 0 \Rightarrow c_2 = -1$

$1 = \frac{-1}{\cancel{2}-1}$ $x = \frac{-1}{z-2-1} = \frac{-1}{z-3}$

$= y$

Math 3335 Homework Solutions

p125 #4 Find the divergence of the field

$$\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$

Is the divergence of this field defined at every point in space?

Solution $$\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{1\cdot(x^2+y^2+z^2)^{3/2} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3}$$

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$$= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly (permute x, y, z) $$\frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} = \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

and $$\frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{3/2}}.$$

So that $$\nabla\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}\right) =$$

$$\frac{(-2x^2+y^2+z^2)+(x^2-2y^2+z^2)+(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{3/2}} = 0.$$

The divergence of this field is not defined at $(0,0,0)$.

Math 3335 Homework Solutions

p125 #10 What can you say about the divergence of the vector field in figure 3.10 at points P, Q, and R? Assume no variation of F in the z direction and that F_3 is identically zero.

Solution From the figure it appears that

$$F = f(y)\,i$$

Then $\operatorname{div} F = \frac{\partial}{\partial x} f(y)\,i = 0.$

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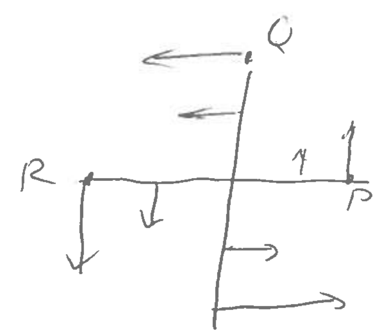
(for example, F might be $(y-2)i$, since $F=0$ on a horizontal line that might be $y=2$).

Math 3335 Homework Solutions

p125 # 11 . What can you say about the divergence of the vector field in figure 3.11 at points P, Q and R? Assume no variation of F in the z direction and that F_3 is zero.



Solution

At P,

$F_1 = 0$ along x - axis

(8)

So $\frac{\partial F_1}{\partial x} = 0$.

The vector field F has constant length on the circle $x^2+y^2=r^2$, and at P, $\frac{\partial F_2}{\partial y} = \frac{\partial F_2}{\partial \theta}\frac{d\theta}{dy}$

$= \frac{\partial F_2}{\partial \theta}\frac{1}{r}$. Since F_2 attains its maximum value over the circle at P, $\frac{\partial F_2}{\partial \theta} = 0$ at P, so $\frac{\partial F_2}{\partial y} = 0$ at P.

Thus

$$\text{div}(F) = F_{1x} + F_{2y} + F_{3z} = 0+0+0.$$

Similarly $\text{div}(F) = 0$ at Q and at R.

Math 3335 Homework Solutions

Handout problems HW#5

2. $\frac{x^2}{3} = xy + 2yz + xz - 10 \quad (*)$

a. Let $f(x,y,z) = \frac{x^2}{3} - xy - 2yz - xz$

Then $(*)$ is equivalent to $f(x,y,z) = -10$

(6) Check that $f(3,1,2) = 3 - 3 - 4 - 6 = -10$

So $(*)$ describes the level set of f that contains $(3,1,2)$.

b. $\nabla f(x,y,z) = \left(\frac{2x}{3} - y - z\right)i - (x+2z)j - (2y+x)k$

(8) So $\nabla f(3,1,2) = (2-1-2)i - (3+4)j - 5k$

$= -i - 7j - 5k$

Then the equation of the tangent plane to the level set is $(-1)(x-3) - 7(y-1) - 5(z-2) = 0$. (1)

c. If $z = g(x,y)$ is defined implicitly by $(*)$ in a neighborhood

(10) of $(3,1,2)$, then the graph of g has the tangent plane

at $(3,1,2)$: $z - 2 = \frac{\partial g}{\partial x}(x-3) + \frac{\partial g}{\partial y}(y-1)$

If we compare this to (1): $z - 2 = -\frac{1}{5}(x-3) - \frac{7}{5}(y-1)$

we find $\frac{\partial z}{\partial x}(3,1) = -1/5$

d. If $x = h(y,z)$ is defined implicitly by $(*)$ near $(3,1,2)$,

(10) then the graph of h has the tangent plane at $(3,1,2)$:

$x - 3 = \frac{\partial h}{\partial y}\cdot(y-1) + \frac{\partial h}{\partial z}(z-2)$

(1) gives $x - 3 = -7(y-1) - 5(z-2)$

So $\frac{\partial x}{\partial y}(1,2) = -7$.