Math 3335 Homework Solutions.

P132 #9 Can you find a vector field whose curl is $y\mathbf{i}$? $x\mathbf{i}$?

Solution: If $F = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and curl $F = y\mathbf{i}$,

then $\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} = y \quad \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} = 0, \ \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$

these equations are satisfied if $F_3 = y^2/2$, $F_2 = F_1 = 0$.

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If $\nabla \times F = x\mathbf{i}$ then

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = x, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

I can't find a vector field F that satisfies $\nabla \times F = x\mathbf{i}$

If, for example, $F_3 = xy$ and $F_2 = g(x,y)$, then

$\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} = x$, but $\dfrac{\partial F_3}{\partial x} = y$ so $\dfrac{\partial F_1}{\partial z} = y \Rightarrow F_1 = yz + f(x,y)$

$\Rightarrow \dfrac{\partial F_1}{\partial y} = z + \dfrac{\partial f}{\partial y} = \dfrac{\partial F_2}{\partial x} = \dfrac{\partial g}{\partial x}(x,y) \Rightarrow z + \dfrac{\partial f}{\partial y}(x,y) = \dfrac{\partial g}{\partial x}(x,y)$

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or $z = \dfrac{\partial g}{\partial x}(x,y) - \dfrac{\partial f}{\partial y}(x,y)$ which does not depend on z.

Looking ahead, we note that $\nabla \cdot (\nabla \times F) = 0$ but $\nabla \cdot (x\mathbf{i}) = 1 \neq 0$.

So $x\mathbf{i}$ cannot be $\nabla \times F$.

10. $F = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$

a. $\nabla \times F = \mathbf{i}(\partial_y x - \partial_z z^2) - \mathbf{j}(\partial_x x - \partial_z y^2) + \mathbf{k}(\partial_x z^2 - \partial_y y^2)$

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$= -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}$

b. $R(t) = \cos \pi t\,\mathbf{i} + \sin \pi t\,\mathbf{j} + t^2\mathbf{k}$, Find $(\nabla \times F) \cdot \dfrac{R'(t)}{|R'(t)|}$ at $t = 1$

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$R(1) = -\mathbf{i} + 0\mathbf{j} + \mathbf{k} \quad x = -1, y = 0, z = 1$

$R'(t) = -\pi \sin(\pi t)\mathbf{i} + \pi \cos \pi t\,\mathbf{j} + 2t\mathbf{k}$

$R'(1) = 0 - \pi\mathbf{j} + 2\mathbf{k} \quad |R'(1)| = \sqrt{\pi^2 + 4}$

$(\nabla \times F)(-1, 0, 1) = -2\mathbf{i} - \mathbf{j} + 0\mathbf{k}$

Then $(\nabla \times F)(-1, 0, 1) \cdot \dfrac{R'(1)}{|R'(1)|} = (-2\mathbf{i} - \mathbf{j}) \cdot \dfrac{(-\pi\mathbf{j} + 2\mathbf{k})}{\sqrt{\pi^2+4}} = \dfrac{\pi}{\sqrt{\pi^2+4}}$

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p132 #11. Let $\vec{F}(x,y,z)$ be defined in all space, and consider an intelligent ant living on the xy plane. Suppose all the ant knows about F is its values on the xy plane.

(a). Can this ant compute curl F? Explain briefly

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Solution No. The ant cannot compute z-derivatives. So it cannot compute $\frac{\partial F_1}{\partial z}$ or $\frac{\partial F_2}{\partial z}$, which are needed for the j and i components of curl F.

(b) Can this ant compute (curl F)$\cdot$k? Explain briefly.

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Solution Yes (smart ant!). $(\text{curl} F)\cdot k = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ which can be computed ~~tes~~ (on $z=0$) using just values of F_2 and F_1 on the xy plane.

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p 132 # 12 Find curl $[f(R)\vec{R})$ where $\vec{R} = xi + yj + zk$, and $R = |R|$ and f is a differentiable function.

(a) by direct calculation.

Solution First note that $\frac{\partial}{\partial x} f(R) = f'(R)\frac{x}{R}$

Similarly $\frac{\partial}{\partial y} f(R) = f'(R)\frac{y}{R}$

and $\frac{\partial}{\partial z} f(R) = f'(R)\frac{z}{R}$.

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So $\text{curl}[f(R)\vec{R}] = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(R)x & f(R)y & f(R)z \end{vmatrix}$

$$= i\left(f'(R)\frac{yz}{R} - f'(R)\frac{zy}{R}\right)$$

$$- j\left(f'(R)\frac{xz}{R} - f'(R)\frac{zx}{R}\right) + k\left(f'(R)\frac{xy}{R} - f'(R)\frac{yx}{R}\right)$$

$$= \vec{0}.$$

(b). by geometrical interpretation.

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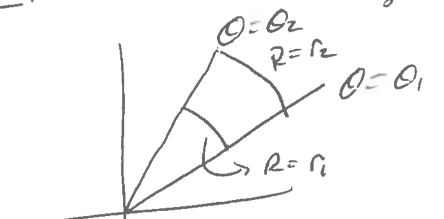
Solution Consider a wedge in the x-y plane:



The integral of $F \cdot ds$ around the boundary of this wedge is:

$$\int_{r_1}^{r_2} f(r)(r\cos\theta_1 i + r\sin\theta_1 j)\cdot[\cos\theta_1 i + \sin\theta_1 j]\,dr$$

$$- \int_{r_1}^{r_2} f(r)(r\cos\theta_2 i + r\sin\theta_2 j)\cdot[\cos\theta_2 i + \sin\theta_2 j]\,dr$$

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p132 #12 b cont $\quad + \displaystyle\int_{\theta_1}^{\theta_2} [f(r_2)[r_2\cos\theta\, i + r_2 \sin\theta\, j] \cdot [-\sin\theta\, i + \cos\theta\, j] \cdot r_2\, d\theta$

$$- \int_{\theta_1}^{\theta_2} f(r_1)[r_1\cos\theta\, i + r_1 \sin\theta\, j] \cdot [-\sin\theta\, i + \cos\theta\, j] \cdot r_1\, d\theta$$

$$= \int_{r_1}^{r_2} f(r)\cdot r\, dr - \int_{r_1}^{r_2} f(r)\, r\, dr + 0 + 0 = 0.$$

Similar calculations hold for wedges in the yz-plane or the xz plane, (Planes parallel to the xy, yz, or xz planes may be more difficult).

This at least suggests that curl $F = 0$.

Better geometric interpretations?

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p135 #8 If $f(x,y,z) = xyz + e^{xz}$, find $\nabla \cdot (\nabla f)$.

Solution

$$\nabla f = (yz + ze^{xz})i + xzj + (xy + xe^{xz})k$$

Then

6 $$\nabla \cdot \nabla f = \frac{\partial}{\partial x}(yz + ze^{xz}) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy + xe^{xz})$$

$$= z^2 e^{xz} + 0 + x^2 e^{xz}$$

$$= (x^2 + z^2)e^{xz}.$$

p140 #2 Find $\nabla^2 f$, given that $f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{1/2}}$

Solution $$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

10 $$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}x(x^2+y^2+z^2)^{-5/2} \cdot 2x$$

$$= \frac{3x^2 - (x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly $$\frac{\partial^2 f}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}}$$

So $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{2x^2 - 2x^2 + 2y^2 - 2y^2 + 2z^2 - 2z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= 0$$

Once we learn ∇^2 in spherical coordinates this should be easier.