Math 3335 Homework Solutions

HW #7

p 150 # 6 If $A = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is a constant vector

$R = x\vec{i} + y\vec{j} + z\vec{k}$, show

$$\nabla \cdot \left(\frac{A \times R}{|R|} \right) = 0$$

Solution By (3.28) this equals $\left(\nabla \frac{1}{|R|}\right) \cdot (A \times R)$

$+ \frac{1}{|R|} \nabla \cdot (A \times R)$

(8) $= \underset{(p.\,150)}{\frac{-R}{|R|^3} \cdot (A \times R)} + \frac{1}{|R|} \underset{(3.36)}{\left[R \cdot (\nabla \times A) - A \cdot (\nabla \times R) \right)}$

Since $\nabla \times A = 0$ and $R = \nabla \frac{|R|^2}{2}$, this is 0. $(R \cdot (A \times R) = 0)$ (3.40)

#7 $V(R) = Af(R \cdot B)$, A and B are constants, show

curl V $\perp$ A and $\perp$ B.

(8) Solution By (3.29), $\nabla \times V = \nabla f(R \cdot B) \times A + (\nabla \times A) f(R \cdot B)$

$\nabla \times A = 0$ and $\nabla f(R \cdot B) = f'(R \cdot B) B$

So $\nabla \times V = f'(R \cdot B) B \times A \perp A$ and B.

#10 a. A const, find $\nabla \cdot (|R|^2 A)$

(8) Solution $= (\nabla |R|^2) \cdot A + |R|^2 \nabla \cdot A$

$= 2R \cdot A + 0$

b. $\nabla \times (A \times R) = (R \cdot \nabla) A - (A \cdot \nabla) R + (\nabla \cdot R) A - (\nabla \cdot A) R$

(8) A constant $\Rightarrow 0 - (a_1 \partial_x (xi + yj + zk) + a_2 \partial_x (xi + yj + zk) + a_3 \partial_z (xi + yj + zk))$

$+ 3A$

$= -A + 3A = 2A$

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p169 #3   Verify eq 3.60

$$\operatorname{curl} F = \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \partial_\rho & \partial_\theta & \partial_z \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

Solution

Right hand side =

$$\frac{1}{\rho}\left(e_\rho\left(\frac{\partial F_z}{\partial \theta} - \frac{\partial(\rho F_\theta)}{\partial z}\right) - \rho e_\theta\left(\frac{\partial F_z}{\partial \rho} - \frac{\partial F_\rho}{\partial z}\right) + e_z\left(\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}\right)\right)$$

8

$$= \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \theta} - \frac{\rho}{\rho}\frac{\partial F_\theta}{\partial z}\right)e_\rho + \frac{\rho}{\rho}\left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)e_\theta + \frac{1}{\rho}\left(\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta}\right)e_z$$

$= \operatorname{curl} \mathbf{F}$.

#5   Verify eq. (3.68)

8

$$\operatorname{curl} F = \frac{1}{r^2 \sin\phi}\begin{vmatrix} e_r & r e_\phi & r\sin\phi\, e_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ F_r & r F_\phi & r\sin\phi\, F_\theta \end{vmatrix}$$

Solution $= \frac{1}{r^2\sin\phi}\left(e_r\left(\frac{\partial}{\partial\phi}(r\sin\phi\, F_\theta) - \frac{\partial}{\partial\theta}(r F_\phi)\right)\right.$

$- r e_\phi\left(\frac{\partial}{\partial r}(r\sin\phi\, F_\theta) - \frac{\partial}{\partial\theta}F_r\right)$

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p169 #5 cont'd $\quad + r\sin\phi\, e_\phi\left(\frac{\partial}{\partial r}(rF_\phi) - \frac{\partial}{\partial \phi}F_r\right)\Big)$

$$= \frac{1}{r\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right)e_r \qquad \left(\frac{r}{r}=1\right)$$

$$+ \frac{r}{r^2\sin\phi}\left(\frac{\partial F_r}{\partial\theta} - \frac{\partial}{\partial r}(r\sin\phi\, F_\theta)\right)e_\phi$$

$$+ \frac{r\sin\phi}{r^2\sin\phi}\left(\frac{\partial}{\partial r}(rF_\phi) - \frac{\partial}{\partial\phi}F_r\right)e_\theta$$

$$= \text{curl}\, F.$$

#7 Show that if f is a function of r only, then

$$\nabla^2 f = f''(r) + \frac{2}{r}f'(r).$$

Solution $\nabla f = \frac{\partial f}{\partial r}e_r = f'(r)e_r.$

(6)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f'(r)\right) + 0$$

$$= f''(r) + \frac{2r}{r^2}f'(r) = f''(r) + \frac{2}{r}f'(r).$$

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p169 # 9 What is the arc length of the curve
$r = \sin\phi \quad \theta = \pi/2$, for $0 \le \phi \le \pi$?

Solution Since $ds = \sqrt{dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi \, d\theta^2}$

6 $$L = \int_0^{\pi} ds = \int_0^{\pi} \sqrt{dr^2 + r^2 d\phi^2} \qquad d\theta = 0$$

$$dr = \cos\phi \, d\phi \;, \; r = \sin\phi$$

$$L = \int_0^{\pi} \sqrt{\cos^2\phi + \sin^2\phi} \, d\phi = \pi .$$
$r = \sin\phi$

#12 Compute the divergence and curl, in spherical coordinates, of $\vec{F}(r,\phi,\theta) = e_r + r e_\phi + r\cos\phi \, e_\theta$.

10 Solution $$\nabla \cdot F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot 1) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}(r\cos\phi) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(r\cdot\sin\phi)$$

(Since $F_r = 1 \quad F_\phi = 1, \; F_\theta = r\cos\phi$)

$$= \frac{2}{r} + 0 + \tan\phi$$

$$\nabla \times F = \frac{1}{r^2\sin\phi}\begin{vmatrix} e_r & r e_\phi & r\sin\phi \, e_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ 1 & r^2 & r^2\sin\phi\cos\phi \end{vmatrix}$$

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p164 # 12, cont

$$= \frac{1}{r^3 \sin\phi}\Big(e_r\big(r^2(\cos^2\phi - \sin^2\phi)\big) - r e_\phi\,(2r\sin\phi\cos\phi)$$

$$+ r\sin\phi\, e_\theta\,(2r)\Big)$$

$$= \frac{\cos 2\phi}{\sin\phi} e_r - 2\cos\phi\, e_\phi + 2 e_\theta$$