Math 3335 Homework Solutions

p190 #3 Find $\int F \cdot dR$ from $(1,0,0)$ to $(1,0,4)$, if $F = xi - yj + zk$

a. along the line segment joining $(1,0,0)$ to $(1,0,4)$.

Solution $r(t) = (1,0,0) + t(0,0,4)$
$= i + 4kt \quad \underline{0 \le t \le 1}$

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$r'(t) = 4k$

$$\int F \cdot dR = \int_0^1 (1i - 0j + 4tk) \cdot 4k \, dt$$

$$= \int_0^1 16t \, dt = \frac{16t^2}{2}\Big|_0^1 = 8.$$

b. along the helix $x = \cos(2\pi t)$, $y = \sin(2\pi t)$, $z = 4t$. $0 \le t \le 1$.

Solution $r'(t) = -2\pi \sin(2\pi t) i + 2\pi \cos(2\pi t) j + 4k$

$$\int F \cdot dR = \int_0^1 \cos(2\pi t)(-2\pi \sin(2\pi t)) - \sin(2\pi t) \cdot 2\pi \cos(2\pi t) + 4t \cdot 4 \, dt$$

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$$= \int_0^1 -4\pi \cos(2\pi t)\sin(2\pi t) + 16t \, dt$$

$$= \int_0^0 -2u \, du + 8t^2\Big|_0^1 = 8$$

$u = \sin 2\pi t$
$du = 2\pi \cos(2\pi t)$

Note that F is $\nabla\left(\frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2}\right)$ so the integral is independent of path

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p190 #6 Find the integral $\oint \mathbf{F}\cdot d\mathbf{R}$ around the circumference of the circle $x^2 - 2x + y^2 = 2$, $z = 1$, where $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + xyz^2\,\mathbf{k}$.

Solution Rewrite the equation of the circle to recognize its center and radius:

$$(x-1)^2 + y^2 = 3, \quad z = 1$$

center $(x, y, z) = (1, 0, 1)$.

radius $\sqrt{3}$, in plane $z = 1$.

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So parameterize the circle as

$$r(t) = (\sqrt{3}\cos t + 1)\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \mathbf{k}$$

$$x = \sqrt{3}\cos t + 1 \qquad y = \sqrt{3}\sin t \qquad z = 1.$$

$$r'(t) = -\sqrt{3}\sin t\,\mathbf{i} + \sqrt{3}\cos t\,\mathbf{j}.$$

$$\oint \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi} (\sqrt{3}\sin t)(-\sqrt{3}\sin t) + (\sqrt{3}\cos t + 1)\sqrt{3}\cos t\,dt$$

$$= \int_0^{2\pi} -3(\sin^2 t - \cos^2 t) - \sqrt{3}\cos t\,dt$$

$$= -3\int_0^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2t\right)dt + 3\int_0^{2\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos 2t\,dt - \sqrt{3}\int_0^{2\pi} \cos t\,dt$$

$$= -\tfrac{3}{2}2\pi + \tfrac{3}{2}\cdot 2\pi + 0 = 0$$

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p190 # 13 Find $\int R \cdot dR$ from $(1,2,2)$ to $(3,6,6)$ along the line segment joining these points,

(a) In the manner described in the text.

Solution $R(t) = (1,2,2)(1-t) + t(3,6,6) \quad 0 \le t \le 1$

$= i + 2j + 2k + t(2i + 4j + 4k)$

$R'(t) = 2i + 4j + 4k.$

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$$\int R \cdot dR = \int_0^1 ((i + 2j + 2k) + t(2i + 4j + 4k)) \cdot (2i + 4j + 4k)\, dt$$

$$= \int_0^1 18 + t(36)\, dt = 18t + 18t^2 \Big|_{t=0}^{1} = 36.$$

(b) By observing that $R \cdot dR = s\,ds$, where $s = \sqrt{x^2+y^2+z^2}$ is the distance from the origin, and computing $\int_3^9 s\,ds$.

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Solution. $R \cdot \frac{dR}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}$

$$= \frac{d}{dt}\left(\frac{x^2+y^2+z^2}{2}\right) = \frac{d}{dt}(s^2/2) = s\frac{ds}{dt}.$$

Then $\int R \cdot dR = \int_{s=3}^{9} s\,ds$ since $\|(1,2,2)\| = 3$, $\|(3,6,6)\| = 9$

$$= s^2/2 \Big|_3^9 = 81/2 - 9/2 = 72/2 = \underline{36}$$

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p 196 # 9 The region in the plane between two concentric circles

Answer. This region is a domain. If the

two circles are not included the set is open, and it is connected. It is not simply connected because any circle with the same center as the 2 boundary circle and radius between the two radii cannot be contracted to a point without leaving the region.

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10. The region in space between two concentric spheres

Answer This region is a simply connected domain.

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Math 3335 Homework Solutions.

p 204 #2c. Using the method of example 4.4, or some similar method, show that the following fields are not conservative.

$$F = y\vec{i} + x\vec{j} + x^2 k$$ [Suggestion: Consider two paths extending from $(0,0,0)$ to $(1,1,1)$].

<u>Solution</u> Let C be the line segment from $(0,0,0)$ to $(1,1,1)$, parameterized by

$$r(t) = t(i+j+k) \quad 0 \le t \le 1.$$
$$r'(t) = i+j+k$$

(8) Then $$\int_C F(r)\cdot dr = \int_0^1 (ti + tj + t^2k)\cdot(i+j+k)\,dt$$

$$= \int_0^1 2t + t^2\,dt = \left(t^2 + t^3/3\right)\Big|_0^1 = \frac{4}{3}.$$

Let D be the union of two line segments D_1; D_2

D_1 = line segment from $(0,0,0)$ to $(0,1,1)$

$$r_1(t) = tj + tk, \quad r_1'(t) = j + k$$

D_2 = line segment from $(0,1,1)$ to $(1,1,1)$

$$r_2(t) = ti + j + k \quad r_2'(t) = i$$

$$\int_D F(r)\cdot dr = \int_{D_1} F(r)\cdot dr + \int_{D_2} F(r)\cdot dr$$

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p 204 #2c, cont.

$$= \int_0^1 (ti + 0j + 0k) \cdot (j + k)\,dt$$

$$+ \int_0^1 (1i + tj + t^2k) \cdot i\,dt$$

$$= 0 + \int_0^1 1\,dt = 1 \neq 4/3.$$

The line integral of F is not independent of path, so F is not conservative.

3c Using methods similar to that of example 4-5, show that the fields of exercise 2 are not conservative.

6 Solution $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & x & x^2 \end{vmatrix}$

$$= i(0) - j(2x) + k(0) = -2xj \neq \vec{0}.$$

So $\frac{\partial}{\partial x}(F_3) \neq \frac{\partial}{\partial z}(F_1)$

So F cannot be a gradient.

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p 204 # 4 Compute $\oint_C F(r)\cdot dR$ around the closed path consisting of a circle of radius r, centered at the origin, in the xy plane, taking

$$F = (-yi + xj)/(x^2+y^2).$$

(Hint: Change to polar coordinates).

Solution $x = r\cos\theta$ $y = r\sin\theta$ $0 \le \theta \le 2\pi$

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$$\oint_C F(r)\cdot dr = \int_0^{2\pi} \frac{(-r\sin\theta i + r\cos\theta j)}{r^2}\cdot(-r\sin\theta i + r\cos\theta j)\, d\theta$$

$$= \int_0^{2\pi} 1\, d\theta = 2\pi$$

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p 204 #6 Find a potential for the force field
$F = (y + z\cos(xz))\,i + x\,j + (x\cos(xz))\,k$

Solution $\phi(x,y,z) = \int_C F(r)\cdot dr$ where

C is the union of 3 line segments

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C_1 : $(0,0,0)$ to $(x,0,0)$ $r_1(t) = t\,i,\ 0 \le t \le x$.

C_2 $(x,0,0)$ to $(x,y,0)$ $r_2(t) = x\,i + t\,j$
$0 \le t \le y$.

C_3 $(x,y,0)$ to (x,y,z): $r_3(t) = x\,i + y\,j + t\,k$
$0 \le t \le z$.

Then

$$\phi(x,y,z) = \int_0^x (0\,i)\cdot i\,dt + \int_0^y [(t+0)\,i + x\,j + x\cdot 1\,k]\cdot j\,dt$$

$$+ \int_0^z [[y + t\cos(xt)]\,i + x\,j + (x\cos(xt))\,k]\cdot k\,dt$$

$$= 0 + \int_0^y x\,dt + \int_0^z x\cos(xt)\,dt$$

$$= xy + \sin(xt)\Big|_0^z = xy + \sin(xz).$$

Check: $\nabla\phi = (y + z\cos(xz))\,i + x\,j + x\cos(xz)\,k$ ✓