Math 3338 Homework Solutions

Ch. 5, p 212 #1 X has pdf. $f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

a. What is c?

4 Solution $1 = \int_{\mathbb{R}} f(x)\,dx = c\int_{-1}^{1}(1-x^2)\,dx = c\left(x - x^3/3\right)\Big|_{-1}^{1}$

$= c\cdot\left(2/3 + 2/3\right) = \frac{4}{3}c$. So $c = 3/4$.

b. What is the cdf of X?

6 Solution: If $x \le -1$, $F(x) = 0$

If $-1 < x \le 1$, $F(x) = \int_{-1}^{x} 3/4\,(1-t^2)\,dt$

$= \frac{3}{4}\left(t - 1^3/3\right)\Big|_{t=-1}^{x} = 3/4\,(x - x^3/3 + 2/3)$

If $x > 1$, $F(x) = 1$.

14 X uniform on $(0,1)$. Compute $E(X^n)$ using Prop 2.1, check using def'n of expectation

(6) (14) Solution By Prop 2.1, $E(X^n) = \int_0^1 x^n\,dx = \frac{1}{n+1}$

To compute $E(X^n)$ from the definition of expectation we need a pdf for X^n.

Let F be a cdf for X: $F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$

Let G be a cdf for $Y = X^n$:

$G(y) = P(X^n \le y) = P(0 \le x \le y^{1/n}) = \begin{cases} 0 & y^{1/n} \le 0 \\ y^{1/n} & 0 < y^{1/n} \le 1 \\ 1 & y^{1/n} > 1 \end{cases}$

So the pdf for Y is

$g(y) = G'(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{n}y^{\frac{1}{n}-1} & 0 < y \le 1 \\ 0 & y > 1 \end{cases}$

So $E(X^n) = E(Y) = \int_0^1 y\cdot\frac{1}{n}y^{\frac{1}{n}-1}\,dy = \int_0^1 \frac{1}{n}y^{\frac{1}{n}}\,dy = \frac{1}{n}\cdot\frac{1}{\frac{1}{n}+1}\,y^{\frac{1}{n}+1}\Big|_0^1$

$= \frac{1}{1+n}$ ✓