Math 3338 Homework Solutions

p 271 #1 Two fair dice rolled, Find joint pmf for X and Y if

a. X = largest value of the 2 dice, Y = sum of 2 dice

Solution Possible values of X, Y: $Y \geq X+1$, $X \geq Y-X$.

$(X,Y) = (1,2), (2,3), (2,4), (3,4), (3,5), (3,6), (4,5), (4,6), (4,7), (4,8)$

$(5,6), (5,7), (5,8), (5,9), (5,10), (6,7), (6,8), (6,9), (6,10), (6,11), (6,12)$

The sample space is the set of ordered pairs (n,m); $n, m = 1,2,3,4,5,6$

(10) Each of these ordered pairs is equally likely, $P(n,m) = \frac{1}{36}$

There is one way to obtain $(X,Y) = (1,2)$: $n = m = 1$.

So $P((X,Y) = (1,2)) = \frac{1}{36}$

Similarly $P((X,Y) = (2,4)) = P((X,Y) = (3,6)) = P((X,Y) = (4,8)) = P(X,Y) = (5,10))$

$= P((X,Y) = (6,12)) = 1/36$

For all other possible values of (X,Y), there are 2 ways

die 1 = X, die 2 = Y-X, and die 2 = X, die 1 = Y-X

So these probabilities are $\frac{2}{36} = \frac{1}{18}$.

b. X is the value of the first die and Y is the larger

Solution Possible values: $Y \geq X$.

(10) $(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6),$

$(3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)$

For (x,y) such that $y > x$, y = value of 2nd die. There is only one

point in the sample space with this event: $(n,m) = (x,y)$

So if $y > x$, $P((X,Y) = (x,y)) = 1/36$.

If $y = x$, then the second value could be any integer j, $1 \leq j \leq x$.

So $P((X,Y) = (x,x)) = \frac{x}{36}$ $x = 1,2,3,4,5,6$

Math 3338 Homework solutions

p271 #C $\quad$ 5 transistors, 2 defective. Test one at a time, until both defectives ID'd.

N_1 = # tests to ID 1^{st} defective

N_2 = # <u>additional</u> tests to ID 2^{nd} defective.

Find joint pmf for N_1, N_2.

<u>Solution</u>. Possible values for (N_1, N_2) $\quad 1 \le N_1 \le 5, \; 1 \le N_2 \le 5 - N_2$.

$(1,1), (1,2), (1,3), (1,4), (2,1), (2,3), (3,1), (3,2), (4,1)$.

$P((1,1)) = P(2^{nd}\text{def.} \mid 1^{st}\text{def}) \cdot P(1^{st}\text{def})$

$= \left(\frac{1}{4}\right) \cdot \left(\frac{2}{5}\right) = \frac{1}{10}$

(12)

$P(1,2) = P(1^{st}\text{def and } 2^{nd}\text{ok. and } 3^{rd}\text{def})$

$= P(3^{rd}\text{def} \mid 2^{nd}\text{ok and } 1^{st}\text{def}) \cdot P(2^{nd}\text{ ok and } 1^{st}\text{def})$

$= \frac{1}{3} \cdot P(2^{nd}\text{def} \mid 1^{st}\text{ok}) \cdot P(1^{st}\text{ok})$

$= \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{2}{5} = \frac{1}{10}$.

Similarly $P((1,3)) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \frac{2}{5} = \frac{1}{10}$

$P((1,4)) = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \frac{2}{5} = \frac{1}{10}$

$P(2,1) = \frac{1}{3} \cdot \frac{2}{4} \frac{3}{5} = \frac{1}{10}$

$P(2,2) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \frac{3}{5} = \frac{1}{10}$

$P(2,3) = \frac{1}{1} \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} = \frac{1}{10}$

$P(3,1) = \frac{1}{2} \frac{2}{3} \cdot \frac{2}{4} \frac{3}{5} = \frac{1}{10}$

$P(3,2) = \frac{1}{1} \frac{1}{2} \frac{2}{3} \frac{2}{4} \frac{3}{5} = \frac{1}{10}$

$P(4,1) = \frac{1}{1} \frac{2}{2} \cdot \frac{1}{3} \frac{2}{4} \cdot \frac{3}{5} = \frac{1}{10}$.

Math 3338 Homework Solutions.

p 271 #14 Ambulance located at X, $0 \le x \le L$

Accident at Y, $0 \le Y \le L$.

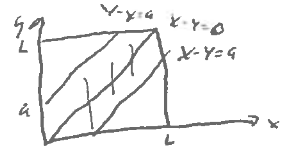
X and Y are independent, both uniformly distributed on $(0, L)$

Compute dist for $|X-Y|$.

Solution X and Y have joint pdf

$$f(x,y) = \begin{cases} \frac{1}{L^2} & 0 \le x \le L, \ 0 \le y \le L \\ 0 & \text{otherwise} \end{cases} \quad \left(= f_X(x)\, f_Y(y)\right)$$

(12)



$P(|X-Y| < a) = ?$

$= \left(L^2 - \text{Area of 2 triangles}\right)\left(\frac{1}{L^2}\right)$

$= 1 - \frac{(L-a)^2}{2} \cdot 2 \cdot \frac{1}{L^2}$

$= 1 - (L-a)^2 \cdot \frac{1}{L^2} = 1 - \frac{L^2}{L^2} + \frac{2La}{L^2} - \frac{a^2}{L^2} = \frac{2a}{L} - \frac{a^2}{L^2}$

Then the pdf for $|X-Y|$ = $\frac{d}{da}\left(\frac{2a}{L} - \frac{a^2}{L^2}\right) = \frac{2}{L} - \frac{2a}{L^2} = \frac{2(L-a)}{L^2}$ $(0 < a < L$, 0 otherwise

18 X, Y indep, X uniform on $(0, L/2)$; Y uniform on $(\frac{L}{2}, L)$.

Find $P(|X-Y| > L/3)$

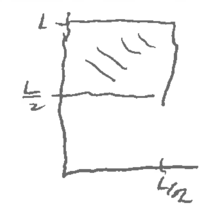
Solution X and Y have a joint distribution with pdf $f(x,y) = f_X(x)\, f_Y(y)$

(12)

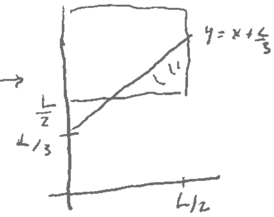
Their joint pdf is constant on the square $0 < x < \frac{L}{2}$, $\frac{L}{2} < y < L$.

The area of this square is $(\frac{L}{2})^2$.

So $f(x,y) = \begin{cases} \frac{4}{L^2} & 0 < x < \frac{L}{2}, \ \frac{L}{2} < y < L \\ 0 & \text{otherwise} \end{cases}$



The intersection of this square with $|x-y| < L/3$ is depicted here:



$P(|X-Y| < \frac{L}{3}) = \int_{L/3}^{L/2} \int_{L/2}^{x+\frac{L}{3}} \frac{4}{L^2}\, dy\, dx$

$= \frac{4}{L^2} \cdot (\text{area of triangle})$

$= \frac{4}{L^2} \cdot (\frac{L}{3})^2 = \frac{4}{9}$

Math 3338 Homework Solutions

p. 271 #20 X and Y have joint pdf $f(x,y) = \begin{cases} xe^{-(x+y)} & x>0, y>0 \\ 0 & \text{other} \end{cases}$

a. are X and Y independent?

Yes $f(x,y) = (xe^{-x})\times e^{-y}$ $x>0, y>0$

Let $f_X(x) = \begin{cases} xe^{-x}, & x>0 \\ 0 & x<0 \end{cases}$ $f_Y(y) = \begin{cases} e^{-y} & y>0 \\ 0 & y<0 \end{cases}$

8

Then $f(x,y) = f_X(x) \cdot f_Y(y)$

So X and Y are indep.

b. If $f(x,y) = \begin{cases} 2 & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$ are X and Y indep?

4

No The set where $f(x,y)>0$ is not a rectangle.