Math 3363 Homework Solutions

9.3.9 Consider $\frac{d^2u}{dx^2} + u = f(x)$, $u(0) = u(L) = 0$.

Assume that $\left(\frac{n\pi}{L}\right)^2 \neq 1$ for all n ($L \neq n\pi$).

a. Solve by variation of parameters

Solution First, solve the homogeneous equation $\frac{d^2u}{dx^2} + u = 0$

$\rightarrow u(x) = A\sin(x) + B\sin(L-x)$

Next 'vary the parameters' A and $B \rightarrow A(x), B(x)$.

Seek $u_p(x) = A(x)\sin x + B(x)\sin(L-x)$

$u_p'(x) = A'(x)\sin(x) + B'(x)\sin(L-x) + A(x)\cos x - B(x)\cos(L-x)$

Assume $A'(x)\sin x + B'(x)\sin(L-x) = 0$

Then $u_p''(x) + u_p(x) = A'(x)\cos x - B'(x)\cos(L-x) = f(x)$

So
$$\begin{pmatrix} \sin x & \sin(L-x) \\ \cos x & -\cos(L-x) \end{pmatrix} \begin{pmatrix} A'(x) \\ B'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

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Cramer's Rule $\Rightarrow$
$$A'(x) = \frac{\det\begin{pmatrix} 0 & \sin(L-x) \\ f(x) & -\cos(L-x) \end{pmatrix}}{\det\begin{pmatrix} \sin x & \sin(L-x) \\ \cos x & -\cos(L-x) \end{pmatrix}} = \frac{-f(x)\sin(L-x)}{-\sin x\cos(L-x) - \cos x\sin(L-x)}$$
$$= \frac{f(x)\sin(L-x)}{\sin(L)}$$

$$B'(x) = \frac{\det\begin{pmatrix} \sin x & 0 \\ \cos x & f(x) \end{pmatrix}}{-\sin L} = -\frac{f(x)\sin x}{\sin(L)}$$

Note that $u_p(x)$ will satisfy the BCs if $A(L)=0$, $B(0)=0$.

So let $A(x) = -\int_x^L f(t)\frac{\sin(L-t)}{\sin L}\,dt$, $B(x) = -\int_0^x f(t)\frac{\sin(t)}{\sin L}\,dt$

Then
$$u_p(x) = \int_0^x f(t)\left(-\frac{\sin(t)\sin(L-x)}{\sin(L)}\right)dt$$
$$+ \int_x^L f(t)\left(-\frac{\sin(L-t)\sin(x)}{\sin(L)}\right)dt = \int_0^L f(t)\,G(t,x)\,dt$$

where
$$G(t,x) = \begin{cases} -\dfrac{\sin(L-t)\sin(x)}{\sin L} & x \leq t \leq L \\[2ex] -\dfrac{\sin(t)\sin(L-x)}{\sin(L)} & 0 \leq t \leq x \end{cases}$$

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9.3.10 Redo 9.3.9 using eigenfunction expansions.

Solution

1. Solve for eigenfunctions, eigenvalues

$$\frac{d^2u}{dx^2} + u + \lambda u = 0 \qquad u(0) = u(L) = 0$$

$$\Leftrightarrow \frac{d^2u}{dx^2} + (1+\lambda)u = 0 \quad \rightarrow u = \sin\left(\frac{n\pi x}{L}\right), \quad 1+\lambda = \left(\frac{n\pi}{L}\right)^2, \quad \lambda = \left(\frac{n\pi}{L}\right)^2 - 1$$

$$f(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right) \qquad B_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$$u = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \quad A_n = \frac{2}{L}\int_0^L u(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$u'' + u = \sum_{n=1}^{\infty} A_n\left(-\left(\frac{n\pi}{L}\right)^2 + 1\right)\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

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then $A_n = B_n \dfrac{1}{1-\left(\frac{n\pi}{L}\right)^2} = \dfrac{2}{L}\displaystyle\int_0^L f(t)\,\frac{\sin\left(\frac{n\pi t}{L}\right)}{1-\left(\frac{n\pi}{L}\right)^2}\,dt$

$$u(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{L}\int_0^L f(t)\,\frac{\sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{1-\left(\frac{n\pi}{L}\right)^2}\,dt$$

$$= \int_0^L f(t) \sum_{n=1}^{\infty}\frac{2}{L}\cdot\frac{\sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{1-\left(\frac{n\pi}{L}\right)^2}\,dt$$

$$= \int_0^L f(t)\,G(x,t)\,dt$$

$$G(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{1-\left(\frac{n\pi}{L}\right)^2}$$

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9.3.11 Consider $\frac{d^2G}{dx^2} + G = \delta(x-x_0)$, $G(0,x_0) = 0 = G(L,x_0)$ $0 \le x_0 \le L$

a. Solve for this Green's function directly. Why is it necessary to assume $L \neq n\pi$, $n=1,2,\ldots$

Solution $G(x,x_0)$ satisfies $\frac{d^2G}{dx^2} + G = 0$ for $x \neq x_0$

$$G(x,x_0) = A\sin(x) + B\sin(L-x) \quad 0<x<x_0$$
$$= C\sin(x) + D\sin(L-x) \quad x_0<x<L$$

$$G(0,x_0) = B\sin L = 0 \Rightarrow B=0.$$

$$G(L,x_0) = C\sin L = 0 \Rightarrow C=0.$$

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$$G(x_0-,x_0) = A\sin(x_0) = G(x_0+,x_0) = D\sin(L-x_0)$$

$$\rightarrow A = E\sin(L-x_0), \quad D = E\sin(x_0)$$

$$\frac{dG}{dx}(x_0+,x_0) - \frac{dG}{dx}(x_0-,x_0) = 1$$

$$\Rightarrow -D\cos(L-x_0) - A\cos(x_0)$$
$$= -E\sin(x_0)\cos(L-x_0) - E\sin(L-x_0)\cos(x_0)$$
$$= -E\sin(L) = 1 \quad E = \frac{-1}{\sin L}$$

Then $G(x,x_0) = \frac{-1}{\sin L}\begin{cases} \sin(L-x_0)\sin(x) & 0 \le x \le x_0 \le L \\ \sin(x_0)\sin(L-x) & 0 \le x_0 \le x \le L \end{cases}$

If $L = n\pi$ then $\sin(L-x) = \sin(n\pi - x) = \cos(n\pi)\sin(-x) = (-1)^{n+1}\sin(x)$.

So that $\sin x$, $\sin(L-x)$ are linearly dependent.

b. Show that $G(x,x_0) = G(x_0,x)$

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Solution $G(x_0,x) = \frac{-1}{\sin L}\begin{cases} \sin(L-x)\sin x_0 & 0 \le x_0 \le x \le L \\ \sin(x)\sin(L-x_0) & 0 \le x \le x_0 \le L \end{cases}$

$= G(x,x_0)$, above.