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Math 4377 Homework

HW#3

1a. Let $v_1 = \begin{bmatrix}1\\3\\2\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\3\\4\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\0\\2\end{bmatrix}$, $b_1 = \begin{bmatrix}3\\3\\6\end{bmatrix}$, $b_2 = \begin{bmatrix}2\\6\\2\end{bmatrix}$

a. Determine whether b_1 is in Span$\{v_1, v_2, v_3\}$

Solution: b_1 is in the span of $v_1 v_2 v_3$ iff there is a solution to $AX = b_1$, where $A = [v_1\ v_2\ v_3]$

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The augmented matrix is

$$[A\ b_1] = \begin{bmatrix}1 & 2 & 1 & 3\\ 3 & 3 & 0 & 3\\ 2 & 4 & 2 & 6\end{bmatrix} \to \begin{bmatrix}1 & 2 & 1 & 3\\ 0 & -3 & -3 & -6\\ 0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -1\\ 0 & 1 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

p p f

This system is consistent; x_3 is a free variable

$x_1 = -1 + x_3$ $\quad x_2 = 2 - x_3$ $\quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\-1\\1\end{bmatrix} + \begin{bmatrix}-1\\2\\0\end{bmatrix}$

$b_1 \in$ Span$\{v_1, v_2, v_3\}$

b. Is b_2 in Span$\{v_1, v_2, v_3\}$?

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$$[A \mid b_2] = \begin{bmatrix}1 & 2 & 1 & 2\\ 3 & 3 & 0 & 6\\ 2 & 4 & 2 & 2\end{bmatrix} \to \begin{bmatrix}1 & 2 & 1 & 2\\ 0 & -3 & -3 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

This system is inconsistent; b_2 is not in Span$\{v_1, v_2, v_3\}$

c. Find weights x_1, x_2, x_3 such that $b_1 = x_1 v_1 + x_2 v_2 + x_3 v_3$

Solution: See above, use $x_1 = -1$ $x_2 = 2$

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$$(-1)\begin{bmatrix}1\\3\\2\end{bmatrix} + 2\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}-1+4\\-3+6\\-2+8\end{bmatrix} = \begin{bmatrix}3\\3\\6\end{bmatrix} \checkmark$$

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2. Determine which of the following sets are bases for $P_2(\mathbb{R})$.

a. $\{1+4x-x^2,\ 2-2x+18x^2,\ -1+x-9x^2\}$

Solution $P_2(\mathbb{R})$ has a basis $\{1, x, x^2\}$ so the dimension of $P_2(\mathbb{R})$ is 3. If this set is linearly independent (and of size 3) then it must be a basis, otherwise not.

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If $c_1(1+4x-x^2)+c_2(2-2x+18x^2)+c_3(-1+x-9x^2)=0$

then
$$c_1+2c_2-c_3=0 \quad (1)$$
$$4c_1-2c_2+c_3=0 \quad (x)$$
$$-c_1+18c_2-9c_3=0 \quad (x^2)$$
this uses the linear independence of $\{1, x, x^2\}$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & -2 & 1 \\ -1 & 18 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -10 & 5 \\ 0 & 20 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This system has a free variable and hence it has nonzero solutions. So this set is linearly dependent and is not a basis for $P_2(\mathbb{R})$

Alternate: Show that the set does not span $P_2(\mathbb{R})$

$\Rightarrow$ Solve $c_1(1+4x-x^2)+c_2(2-2x+18x^2)+c_3(-1+x-9x^2)=b_1+b_2x+b_3x^2$

$\to$ Augmented matrix
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 4 & -2 & 1 & b_2 \\ -1 & 18 & -9 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & -10 & 5 & b_2-4b_1 \\ 0 & 20 & -10 & b_3+b_1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 2 & -1 & (b_2-4b_1)/(-5) \\ 0 & 0 & 0 & b_3+b_1+2(b_2-4b_1) \end{array}\right]$$

So a solution exists iff $b_3-7b_1+2b_2=0$.

The set is not a basis for $P_2(\mathbb{R})$. It does not span $P_2(\mathbb{R})$.

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2b $\{-1+3x-2x^2,\ 2-4x-5x^2,\ 4-10x-6x^2\}$

Solution Again, since this is a set of 3 vectors in a space of dimension 3, it suffices to determine whether the vectors are linearly independent.

(10)

If $c_1(-1+3x-2x^2)+c_2(2-4x-5x^2)+c_3(4-10x-6x^2)=0$ (for all x)

then equating like terms:

$$-c_1+2c_2+4c_3=0$$
$$3c_1-4c_2-10c_3=0$$
$$-2c_1-5c_2-6c_3=0$$

The coefficient matrix

$$A=\begin{bmatrix}-1&2&4\\3&-4&-10\\-2&-5&-6\end{bmatrix}\to\begin{bmatrix}1&-2&-4\\0&2&2\\0&-9&-14\end{bmatrix}\to\begin{bmatrix}1&-2&-4\\0&1&1\\0&0&-7\end{bmatrix}$$

A has 3 pivots - there are no free variables in the solution to $Ax=0$. (Or solve: $-7x_3=0 \Rightarrow x_3=0$, $x_2=-x_3=0$, $x_1=2x_2+4x_3=0$. The only solution is $c_1=c_2=c_3=0$.

So these vectors are linearly independent and form a basis for P_2.

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2c. $\{1-2x+x^3, -2+3x-x^2, -2-x+6x^2\}$

Solution Again, the # of vectors $=\dim(P_2)=3$

So we check for linear independence.

If $c_1(-1+2x+x^2)+c_2(-2+3x-x^2)+c_3(-2-x+6x^2)=0$

then, equating like terms,

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$$\begin{aligned} -c_1-2c_2-2c_3&=0 \quad (1)\\ 2c_1+3c_2-c_3&=0 \quad (x)\\ c_1-c_2+6c_3&=0 \quad (x^2)\end{aligned}$$

The coefficient matrix

is $A=\begin{bmatrix}-1&-2&-2\\2&3&-1\\1&-1&6\end{bmatrix}\to\begin{bmatrix}1&2&2\\0&-1&-5\\0&-3&4\end{bmatrix}\to\begin{bmatrix}1&2&2\\0&1&5\\0&0&19\end{bmatrix}$

A has 3 pivots so the only solution is $c_1=c_2=c_3=0$

So this set of vectors is a basis for P_2.

3. Find a subset of S which is a basis for the span of S, if

$$S=\left\{\begin{bmatrix}-3\\1\\2\end{bmatrix},\begin{bmatrix}6\\-2\\-4\end{bmatrix},\begin{bmatrix}-1\\2\\5\end{bmatrix},\begin{bmatrix}1\\3\\8\end{bmatrix},\begin{bmatrix}-7\\-1\\-4\end{bmatrix}\right\}$$

Solution The key is to find dependence relations and to use them to eliminate vectors in S. Write the vectors as columns of a matrix and use elementary row operations to find the reduced echelon form of this matrix:

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$$A=\begin{bmatrix}-3&6&-1&1&-7\\1&-2&2&3&-1\\2&-4&5&8&-4\end{bmatrix}\to\begin{bmatrix}1&-2&2&3&-1\\-3&6&-1&1&-7\\2&-4&5&8&-4\end{bmatrix}\to\begin{bmatrix}1&-2&2&3&-1\\0&0&5&10&-10\\0&0&1&2&-2\end{bmatrix}$$

$$\to\begin{bmatrix}1&-2&2&3&-1\\0&0&1&2&-2\\0&0&1&2&-2\end{bmatrix}\to\begin{bmatrix}1&-2&2&3&-1\\0&0&1&2&-2\\0&0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&-2&0&-1&3\\0&0&1&2&-2\\0&0&0&0&0\end{bmatrix}$$

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HW#3 Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Let A have columns $[a_1\ a_2\ a_3\ a_4\ a_5]$

Then $A \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2a_1 + a_2 = 0$ So $a_2 = -2a_1$

$A \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = -a_1 - 2a_3 + a_4 = 0$ So $a_4 = -a_1 + 2a_3$

$A \cdot \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = -3a_1 + 2a_3 + a_5$ So $a_5 = 3a_1 - 2a_3$.

"The free columns are linear combinations of the previous pivot columns"

So a basis for Span$\{a_1, a_2, a_3, a_4, a_5\}$ is $\{a_1, a_3\}$ = pivot columns of A.

4. Let S be the subspace of $M_{3\times 3}(\mathbb{R})$ which consists of all 3×3 real symmetric matrices ($A^T = A$).

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Find a basis for S. What is the dimension of S?

Solution Any 3×3 matrix $A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$ is symmetric.

And any symmetric matrix has this form

$$A = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ d\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

the matrices $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

clearly span S and are linearly independent. So they are a basis for S and $\dim(S) = 6$