Math 4377 Homework Solutions.

HW#9

1. Let $V$ and $W$ be vector spaces, and let $T: V \to W$ be linear. Suppose $\{v_1, \dots, v_p\}$ is a linearly dependent subset of $V$. Prove that $\{T(v_1), \dots, T(v_p)\}$ is a linearly dependent subset of $W$.

(8)

Proof. Since $\{v_1, \dots, v_p\}$ is linearly dependent, there are scalars $c_1, \dots, c_p$ not all zero such that $c_1 v_1 + \dots + c_p v_p = 0$.

Then $T(c_1 v_1 + \dots + c_p v_p) = T(0) = 0$ and $T(c_1 v_1 + \dots + c_p v_p) = c_1 T(v_1) + \dots + c_p T(v_p)$ since $T$ is linear.

So $c_1 T(v_1) + \dots + c_p T(v_p) = 0$, with $c_i$ not all 0.

Thus $\{T(v_1), \dots, T(v_p)\} \subset W$ is a linearly dependent set.

2. If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear and $T(1,1) = (1,2,2)$, $T(2,3) = (2,-1,2)$ find:

(a) $T(1,0)$ and $T(0,1)$

(8)

Solution Solve $\begin{pmatrix}1\\0\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}2\\3\end{pmatrix} \to \begin{pmatrix}1&2&1\\1&3&0\end{pmatrix} \to \begin{pmatrix}1&2&1\\0&1&-1\end{pmatrix} \to \begin{pmatrix}1&0&3\\0&1&-1\end{pmatrix}$

So $\begin{pmatrix}1\\0\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}$. Thus

$T(1,0) = 3T(1,1) - T(2,3) = 3\cdot(1,2,2) - (2,-1,2) = \underline{(1,7,4)}$

Solve $\begin{pmatrix}0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}2\\3\end{pmatrix} \to \begin{pmatrix}1&2&0\\1&3&1\end{pmatrix} \to \begin{pmatrix}1&2&0\\0&1&1\end{pmatrix} \to \begin{pmatrix}1&0&-2\\0&1&1\end{pmatrix}$ $c_1 = -2, c_2 = 1$

$\begin{pmatrix}0\\1\end{pmatrix} = -2\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix}$ So $T(0,1) = -2T(1,1) + T(2,3)$

$= -2(1,2,2) + (2,-1,2) = \underline{(0,-5,-2)}$

b $T(x,y)$ for any real numbers $x$ and $y$.

(4)

Solution $T(x,y) = xT(1,0) + yT(0,1) = x(1,7,4) + y(0,-5,-2) = (x, 7x-5y, 4x-2y)$.