Math 4377 Homework Solutions

HW 5

2. Let $P: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ project P_2 onto Span$\{1, x^2\}$ along $1-x$. Find $[P]_\beta$. ($P(1) = 1$. What is $P(x)$?)

Solution $P(p(x)) = p(x)$ if $p(x) \in R(P)$. So $P(1) = 1$, $P(x^2) = x^2$.

(8) Since $P(1-x) = 0$, $P(1) - P(x) = 0$ or $P(x) = P(1) = 1$.

Then $[P(1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[P(x)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[P(x^2)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So $[P]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Let V be a vector space with finite dimension. Let $T: V \to V$ be linear. Show that if $N(T) = N(T^2)$, then $R(T) \cap N(T) = \{0\}$. Show that, as a consequence, $V = R(T) \oplus N(T)$.

Solution Let $x \in R(T) \cap N(T)$. Then $x = Ty$, and $Tx = T^2 y = 0$.

(16) Since $N(T^2) = N(T)$, $Ty = 0$. Hence $x = 0$. So $R(T) \cap N(T) = 0$.

To show that $V = R(T) \oplus N(T)$, it remains to show that $V = R(T) + N(T)$. Let $\beta = \{v_1, \dots, v_p\}$ be a basis for $R(T)$.

Let $\gamma = \{w_1, \dots, w_q\}$ be a basis for $N(T)$. By Theorem 2.3, $p + q = \dim V$. $R(T) \cap N(T) = \{0\} \Rightarrow \beta \cup \gamma$ is linearly indep.:

If $c_1 v_1 + \dots + c_p v_p + d_1 w_1 + \dots + d_q w_q = 0$ then

$c_1 v_1 + \dots + c_p v_p = -(d_1 w_1 + \dots + d_q w_q)$. Both sides are in

$R(T) \cap N(T) \Rightarrow c_1 v_1 + \dots + c_p v_p = 0 = d_1 w_1 + \dots + d_q w_q$

Since β and γ are indep, $c_i = 0$, $d_j = 0$ all.

Then $\beta \cup \gamma$ is indep and contains $p + q = \dim V$ vectors

So $\beta \cup \gamma$ is a basis for V.

Consequently $V = R(T) + N(T)$

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$\beta = \{1, x, x^2\}$

1. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(p(x)) = x p'(x)$

$S: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $S(p(x)) = \frac{d}{dx}(x p(x))$

a. $[T]_\beta = ?$ $[S]_\beta = ?$

$[T]_\beta$: $T(1) = 0$ $\quad T(x) = x \cdot 1 = x$ $\quad T(x^2) = x \cdot 2x = 2x^2$

So $[T(1)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad [T(x)]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad [T(x^2)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

(8) and so $[T]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$S(1) = \frac{d}{dx}(x) = 1$ $\quad S(x) = \frac{d}{dx}(x^2) = 2x$ $\quad S(x^2) = 3x^2$

$[S(1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad [S(x)]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $\quad [S(x^2)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$$[S]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

b. Is T one-to-one?

(6) Solution No because $T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$.

c. Is T onto?

(6) Solution No, because $R(T)$ has dimension 2.

$(2 = 3 - 1$, $3 = \dim P_2$, $1 = \dim N(T).)$

d. Is S 1to1?

(6) Solution Yes, because $[S_\beta]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow x_1 = x_2 = x_3 = 0$

$\text{Null}([S]_\beta) = \{0\}$ So $N(S) = \{\vec{0}\}$.

e. Is S onto? Yes, because $\dim R(S) = 3 - \dim N(S) = 3 = \dim P_2$

(6)