Math 4377 Homework Solutions

HW #7

1. Let $V = \mathbb{R}^2$ and let $\beta = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Find explicit formulas for vectors in the dual basis β^* for V^*, as in Example 4 p 120.

Solution We seek linear functions f_1, f_2 on $\mathbb{R}^2$

s.t. $f_1(3,2)=1 \quad f_1(1,1)=0$

$f_2(3,2)=0 \quad f_2(1,1)=1$

(10) Let $f_1(x,y) = ax+by \quad f_2(x,y) = cx+dy$

Then $3a+2b=1 \quad a+b=0$

$3c+2d=0 \quad c+d=1$

Or, $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

So $f_1(x,y) = x-y \quad f_2(x,y) = -2x+3y$

and $\beta^* = \{ f_1, f_2 \}$.

2. Define $f \in (\mathbb{R}^2)^*$ by $f(x,y) = x-3y$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (x-2y, 3y)$.

(5) a. Compute $T^t f$

Solution $(T^t f)(x,y) = f(T(x,y)) = f(x-2y, 3y) = (x-2y) - 3\cdot(3y) = x-11y$.

b. Compute $[T^t]_{\beta^*}$, where β is the standard basis for $\mathbb{R}^2$ and $\beta^* = \{ f_1, f_2 \}$ is the dual basis, by finding scalars a, b, c, d s.t. $T^t(f_1) = a f_1 + b f_2, \quad T^t(f_2) = c f_1 + d f_2$

(8) Solution $f_1(x,y) = x \quad f_2(x,y) = y$

$(T^t f_1)(x,y) = f_1(T(x,y)) = f_1(x-2y, 3y) = x-2y = 1\cdot f_1(x,y) - 2 f_2(x,y)$

$(T^t f_2)(x,y) = f_2(T(x,y)) = f_2(x-2y, 3y) = 3y = 0\cdot f_1(x,y) + 3 f_2(x,y)$

So $[T^t]_{\beta^*} = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$.

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2c Compute $[T]_\beta$ and $[T]_\beta^t$ and compare your results with (h).

(5) Solution $T(1,0)=(1,0)$ $T(0,1)=(-2,3)$

So $[T]_\beta = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$, $[T]_\beta^t = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = [T^t]_{\beta^*}$

3. Let
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 2 \end{pmatrix}.$$

Use elementary row operations to reduce A to I_3. Then use the elementary row operations to express A^{-1} as a product of elementary matrices. Then express A as a product of elementary matrices

Solution (1) Replace R_2 by $R_2 - 2R_1$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 2 \end{pmatrix}$ $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(12)

(2) Replace R_3 by $R_3 - 3R_1$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 2 \end{pmatrix}$ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$

(3) Replace R_3 by $R_3 - 5R_2$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$

(4) Multiply R_3 by $1/2$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

Then $E_4 E_3 E_2 E_1 A = I_3$. So $A^{-1} = E_4 E_3 E_2 E_1$

and $A = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$.

$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$ $E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$