Math 4377
February 28, 2019 Exam 1 Review—Chapter 2Name ______________________________

1. Label the following statements as true (T) or false (F).

(a) T F It is impossible for a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ that the kernel $N(T)$ and the range $R(T)$ are the same subspace.
False. Consider $T(x, y) = (y, 0)$

For the following, V and W are vector spaces with finite dimension, $T : V \to W$ is linear and β is a subset of V.

(b) T F If β is linearly dependent then $T(\beta)$ is linearly dependent.
True

(c) T F If $T(\beta)$ is linearly dependent, then β is linearly dependent.
False. Consider $T(x, y) = (y, 0)$ and $\beta = \{(1, 0)\}$.

(In the original review sheet there is no (d).)

(d) T F If V and W have the same finite dimension and $T : V \to W$ is linear and onto, then T is an isomorphism.
True

(e) T F If T and U are linear transformations from V to W and both agree on a linearly independent subset of V, then $T = U$.
False. Consider $\beta = \{(1, 0)\}$, $T(x, y) = (x, y)$, $U(x, y) = (x, 2y)$. T and U agree on the linearly independent set β but are not the same.

(f) T F If β spans V, T and U are linear transformations from V to W and both agree on β, then $T = U$.
True. Note that if β is linearly dependent, T and U must be defined in a way that respects the dependence relations on β.
For j-l, V, W, and Z are vector spaces with finite ordered bases α, β, and γ, respectively, $T : V \to W$ and $U : W \to Z$ are linear.

(g) T F $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$.
False. It should be $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

(h) T F $[T(\mathbf{v})]_\beta = [T]_\alpha^\beta [\mathbf{v}]_\alpha$.
True

(i) T F Assume $V = W$ but α and β are different. If Q is the change of coordinate matrix that changes α coordinates into β coordinates, then $[T]_\beta = Q^{-1} [T]_\alpha Q$.
False It should be $[T]_\alpha = Q^{-1} [T]_\beta Q$, or $[T]_\beta = [T]_\alpha Q^{-1}$.

2. Find the change of coordinates matrix that changes β coordinates into α coordinates, where β and α are the following bases of P_2. (20)

$$\beta = \left\{1,\ x,\ x^2\right\}, \quad \alpha = \left\{1,\ 1+x,\ 1+x+x^2\right\}.$$

The matrix is $[I_{P_2}]_\beta^\alpha = Q$. The first column of Q is $[1]_\alpha = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$. The second column of Q is $[x]_\alpha$. Since $x = 1+x-1$, $[x]_\alpha = \begin{bmatrix}-1 & 1 & 0\end{bmatrix}^T$. The third column of Q is $\left[x^2\right]_\alpha$. Since $x^2 = 1+x+x^2-(1+x)$, $\left[x^2\right]_\alpha = \begin{bmatrix}0 & -1 & 1\end{bmatrix}^T$. So $Q = \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$.

3. Determine whether the following linear transformations are invertible. Justify your answer.

   (a) $T : M_{2\times2} \to M_{2\times2}$ by $T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = \begin{bmatrix}a+b & a\\ c & c+d\end{bmatrix}$. (15)

   Invertible $T^{-1}\left(\begin{bmatrix}e & f\\ g & h\end{bmatrix}\right) = \begin{bmatrix}f & e-f\\ g & h-g\end{bmatrix}$.

   (b) $T : M_{2\times2} \to P_2(\mathbb{R})$ by $T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = a + 2bx + (c+d)x^2$. (15)

   Not invertible The dimension of $M_{2\times2}$ isi 4, and the dimension of $P_2(\mathbb{R})$ is 3. No linear transformation between these two vector spaces can be invertible.