which has no real zeros because, for $0 < \theta < \pi$, the discriminant $4\cos^2\theta - 4$ is negative.

EXERCISES

1. Label the following statements as true or false.
   (a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.
   (b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
   (c) There exists a square matrix with no eigenvectors.
   (d) Eigenvalues must be nonzero scalars.
   (e) Any two eigenvectors are linearly independent.
   (f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
   (g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.
   (h) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.
   (i) Similar matrices always have the same eigenvalues.
   (j) Similar matrices always have the same eigenvectors.
   (k) The sum of two eigenvectors of an operator T is always an eigenvector of T.

2. For each of the following linear operators T on a vector space V and ordered bases β, compute $[\mathsf{T}]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.
   (a) $\mathsf{V} = \mathsf{R}^2$, $\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$
   (b) $\mathsf{V} = \mathsf{P}_1(R)$, $\mathsf{T}(a + bx) = (6a - 6b) + (12a - 11b)x$, and $\beta = \{3 + 4x, 2 + 3x\}$
   (c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}$, and
   $$\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$
   (d) $\mathsf{V} = \mathsf{P}_2(R)$, $\mathsf{T}(a + bx + cx^2) =$
   $$(-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2,$$
   and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$

   (e) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(a + bx + cx^2 + dx^3) =$
   $$-d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3,$$
   and $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$
   (f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$, and
   $$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

3. For each of the following matrices $A \in \mathsf{M}_{n\times n}(F)$,
   (i) Determine all the eigenvalues of A.
   (ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
   (iii) If possible, find a basis for F^n consisting of eigenvectors of A.
   (iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = R$
   (b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = R$
   (c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = C$
   (d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = R$

4. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.
   (a) $\mathsf{V} = \mathsf{R}^2$ and $\mathsf{T}(a, b) = (-2a + 3b, -10a + 9b)$
   (b) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$
   (c) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$
   (d) $\mathsf{V} = \mathsf{P}_1(R)$ and $\mathsf{T}(ax + b) = (-6a + 2b)x + (-6a + b)$
   (e) $\mathsf{V} = \mathsf{P}_2(R)$ and $\mathsf{T}(f(x)) = xf'(x) + f(2)x + f(3)$
   (f) $\mathsf{V} = \mathsf{P}_3(R)$ and $\mathsf{T}(f(x)) = f(x) + f(2)x$
   (g) $\mathsf{V} = \mathsf{P}_3(R)$ and $\mathsf{T}(f(x)) = xf'(x) + f''(x) - f(2)$
   (h) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

(i) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

(j) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}(A) = A^t + 2 \cdot \mathrm{tr}(A) \cdot I_2$

5. Prove Theorem 5.4.

6. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$.

7. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(\mathsf{T})$, as follows: Choose any ordered basis β for V, and define $\det(\mathsf{T}) = \det([\mathsf{T}]_\beta)$.

(a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_\gamma)$.

(b) Prove that T is invertible if and only if $\det(\mathsf{T}) \neq 0$.

(c) Prove that if T is invertible, then $\det(\mathsf{T}^{-1}) = [\det(\mathsf{T})]^{-1}$.

(d) Prove that if U is also a linear operator on V, then $\det(\mathsf{TU}) = \det(\mathsf{T}) \cdot \det(\mathsf{U})$.

(e) Prove that $\det(\mathsf{T} - \lambda \mathsf{I_V}) = \det([\mathsf{T}]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

8. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(c) State and prove results analogous to (a) and (b) for matrices.

9. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

10. Let V be a finite-dimensional vector space, and let λ be any scalar.

(a) For any ordered basis β for V, prove that $[\lambda \mathsf{I_V}]_\beta = \lambda I$.

(b) Compute the characteristic polynomial of $\lambda \mathsf{I_V}$.

(c) Show that $\lambda \mathsf{I_V}$ is diagonalizable and has only one eigenvalue.

11. A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

12. (a) Prove that similar matrices have the same characteristic polynomial.

(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

13. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [\mathsf{T}]_\beta$. In reference to Figure 5.1, prove the following.

(a) If $v \in \mathsf{V}$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to λ.

(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

14.† For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

15.† (a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

(b) State and prove the analogous result for matrices.

16. (a) Prove that similar matrices have the same trace. *Hint:* Use Exercise 13 of Section 2.3.

(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

17. Let T be the linear operator on $\mathsf{M}_{n\times n}(R)$ defined by $\mathsf{T}(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

18. Let $A, B \in \mathsf{M}_{n\times n}(C)$.

(a) Prove that if B is invertible, then there exists a scalar $c \in C$ such that $A + cB$ is not invertible. *Hint:* Examine $\det(A + cB)$.

(b) Find nonzero 2×2 matrices A and B such that both A and $A+cB$ are invertible for all $c \in C$.

19.† Let A and B be similar $n \times n$ matrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and ordered bases β and γ for V such that $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_\gamma$. *Hint:* Use Exercise 14 of Section 2.5.

20. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

21. Let A and $f(t)$ be as in Exercise 20.

(a) Prove that $f(t) = (A_{11}-t)(A_{22}-t)\cdots(A_{nn}-t)+q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$. *Hint:* Apply mathematical induction to n.

(b) Show that $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

22.† (a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(\mathsf{T})(x) = g(\lambda)x$. That is, x is an eigenvector of $g(\mathsf{T})$ with corresponding eigenvalue $g(\lambda)$.

(b) State and prove a comparable result for matrices.

(c) Verify (b) for the matrix A in Exercise 3(a) with polynomial $g(t) = 2t^2 - t + 1$, eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and corresponding eigenvalue $\lambda = 4$.

23. Use Exercise 22 to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(\mathsf{T}) = \mathsf{T}_0$, the zero operator. (In Section 5.4 we prove that this result does not depend on the diagonalizability of T.)

24. Use Exercise 21(a) to prove Theorem 5.3.

25. Prove Corollaries 1 and 2 of Theorem 5.3.

26. Determine the number of distinct characteristic polynomials of matrices in $\mathsf{M}_{2\times 2}(Z_2)$.

5.2 DIAGONALIZABILITY

In Section 5.1, we presented the diagonalization problem and observed that not all linear operators or matrices are diagonalizable. Although we are able to diagonalize operators and matrices and even obtain a necessary and sufficient condition for diagonalizability (Theorem 5.1 p. 246), we have not yet solved the diagonalization problem. What is still needed is a simple test to determine whether an operator or a matrix can be diagonalized, as well as a method for actually finding a basis of eigenvectors. In this section, we develop such a test and method.

In Example 6 of Section 5.1, we obtained a basis of eigenvectors by choosing one eigenvector corresponding to each eigenvalue. In general, such a procedure does not yield a basis, but the following theorem shows that any set constructed in this manner is linearly independent.

Theorem 5.5. *Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \ldots, v_k$ are eigenvectors of T such that λ_i corresponds to v_i ($1 \le i \le k$), then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.*

Proof. The proof is by mathematical induction on k. Suppose that $k=1$. Then $v_1 \neq 0$ since v_1 is an eigenvector, and hence $\{v_1\}$ is linearly independent. Now assume that the theorem holds for $k-1$ distinct eigenvalues, where $k-1 \ge 1$, and that we have k eigenvectors $v_1, v_2, \ldots, v_k$ corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We wish to show that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Suppose that $a_1, a_2, \ldots, a_k$ are scalars such that

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0. \tag{1}$$

Applying $\mathsf{T} - \lambda_k\mathsf{I}$ to both sides of (1), we obtain

$$a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_2 - \lambda_k)v_2 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0.$$

By the induction hypothesis $\{v_1, v_2, \ldots, v_{k-1}\}$ is linearly independent, and hence

$$a_1(\lambda_1 - \lambda_k) = a_2(\lambda_2 - \lambda_k) = \cdots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, it follows that $\lambda_i - \lambda_k \neq 0$ for $1 \le i \le k-1$. So $a_1 = a_2 = \cdots = a_{k-1} = 0$, and (1) therefore reduces to $a_kv_k = 0$. But $v_k \neq 0$ and therefore $a_k = 0$. Consequently $a_1 = a_2 = \cdots = a_k = 0$, and it follows that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. ∎

Corollary. *Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.*