Notes on Complex Numbers

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A complex number z can be represented in the form $z = a + bi$, where $i^2 = -1$. The *conjugate* of z is $\bar{z} = a - bi$. If $w = c + di$, then $z+w = (a+c)+(b+d)i$, and $\overline{z+w} = (a+c)-(b+d)i = \bar{z}+\bar{w}$. Products are more complicated: $zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$. So

$$\overline{zw} = (ac - bd) - (ad + bc)i = \bar{z}\bar{w}. \text{ Check!}$$

Let

$$r = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$$

Then $z = a + bi = r\left(\frac{a}{r} + \frac{b}{r}i\right)$. Note that

$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2 + b^2}{r^2} = 1.$$

Hence the point $P = \left(\frac{a}{r},\ \frac{b}{r}\right)$ lies on the circle $x^2 + y^2 = 1$. Let L be the line from the origin to P, and let θ be the angle from the positive x-axis to the line L. (*Draw yourself a picture!*) Then we must have that $P = (\cos(\theta),\ \sin(\theta))$. Thus

$$z = r\left(\cos(\theta) + i\sin(\theta)\right) = re^{i\theta}.$$

Observe that

$$z^2 = r^2\left(\cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta)\right) \quad (1)$$

$$= r^2\left(\cos(2\theta) + i\sin(2\theta)\right) = r^2e^{i2\theta}. \quad (2)$$

Thus if z has a magnitude of r and an angle of θ, then z^2 has a magnitude of r^2 and an angle of 2θ.

This observation illustrates the general pattern for powers of complex numbers:

$$z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right) = r^ne^{in\theta}.$$

The fact that $(cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$ leads to formulas for $\sin(n\theta)$ and $\cos(n\theta)$. These formulas are known collectively as DeMoivre's formula. (See the Wikipedia article on DeMoivre's formula).

This fact can also be helpful in solving complex polynomial equations. In particular let us look at finding complex n^{th} roots of a complex number. That is, we want to find solutions of the equation

$$z^n = w$$

where z and w are complex.

For example, let us find the cube roots of 1—that is, let us solve the equation $z^3 = 1$. 1 has a magnitude of 1 and and angle of 0. Note, however, that any complex number with a magnitude of 1 and and angle of $2k\pi$ will also be 1, if k is an integer.

If $z = r\left(\cos(\theta) + i\sin(\theta)\right)$, then $z^3 = r^3\left(\cos(3\theta) + i\sin(3\theta)\right)$. This quantity is 1 if

$$r^3 = 1, \quad (r = 1), \tag{3}$$

$$3\theta = 2k\pi \text{ (where } k \text{ is an integer).} \tag{4}$$

Thus, the choices for θ are: $\theta = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$. The corresponding values for z are:

$$z_1 = 1, \tag{5}$$

$$z_2 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \tag{6}$$

$$z_3 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \tag{7}$$

Plot these three points in the plane!

Finding n^{th} roots of other real numbers is just as easy. Try

(1) 5^{th} roots of 32 (five of them),

(2) 4^{th} roots of 81.

Finding n^th roots of complex numbers is just a little more difficult. For example, let us find the square roots of i. $i = \cos(\pi/2) + i\sin(\pi/2)$. Thus, to solve $z^2 = i$, we have

$$z^2 = r^2\left(\cos(2\theta) + i\sin(2\theta)\right) = \cos(\pi/2) + i\sin(\pi/2).$$

This implies that $r = 1$ and $2\theta = \frac{\pi}{2} + 2k\pi$. Solving for θ, we have

$$\theta = \frac{\pi}{4} + k\pi.$$

The choices for θ are $\theta = \frac{\pi}{4},\ \frac{5\pi}{4}$. Thus the two square roots of i are:

$$z_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \frac{\sqrt{2}}{2}\left(1+i\right), \tag{8}$$

$$z_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -\frac{\sqrt{2}}{2}\left(1+i\right). \tag{9}$$

This isn't hard, is it? Try

(1) 4^{th} roots of i.

(2) 3^{rd} roots of $27i$

(3) 3^{rd} roots of $4\sqrt{2}\,(1-i)$.

Can you make up more problem like this? Sure you can!

That's all, folks!