1.2 Quotient and dual spaces

Def. Let V be a vector space and W a vector subspace of V. A

W-coset is a set of the form

$$v + W := \{v + w \mid w \in W\}.$$

- Distinct W-cosets decompose V into a disjoint collection of subsets of V. (check if $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$, otherwise $v_1 + W$ and $v_2 + W$ are disjoint.)
- We write V/W for the set of distinct collection of subsets of V.
- Vector addition on V/W: $(v_1 + W) + (v_2 + W) := (v_1 + v_2) + W$.

  Scalar multiplication on V/W: $\lambda(v + W) := (\lambda v) + W$.

  These operations make V/W a vector space, called the quotient space of V by W.
- Operations on V/W are well-defined. (Check if $v_1 + W = v_1' + W$, $v_2 + W = v_2' + W$, then $(v_1 + v_2) + W = (v_1' + v_2') + W$.)
- A quotient map $\pi : V \to V/W$ is $\pi(v) := v + W$. π is linear and maps V surjectively onto V/W.
- The zero vector in V/W is W, so $v \in \ker(\pi)$ iff $v \in W$.

  Thus, $\underset{\uparrow \atop \text{im } \pi}{\dim(V/W)} = \dim(V) - \underset{\uparrow \atop \ker \pi}{\dim(W)}$
- Prop. Let $A : V \to U$ be a linear map of vector spaces. If $W \subset \ker A$, there exists a unique linear map $A^\# : V/W \to U$ such that $A = A^\# \circ \pi$.

  proof. First we define $A^\#$: for any $u \in V/W$, choose a lift to V (i.e., a vector v such that $\pi(v) = u$). Define $A^\#(u) = A(v)$. This does not depend on the lift because for any other lift v', it has the form $v' = v + w$, thus $A(v') = A(v)$ since $A(w) = 0$.

This is also a linear map since given $u_1, u_2 \in V/W$, $c \in \mathbb{R}$:

$$A^{\#}(u_1 + u_2) = A(v_1 + v_2)$$

since $v_1 + v_2$ is a lift of $u_1 + u_2$. So

$$A(v_1 + v_2) = A(v_1) + A(v_2) = A^{\#}(u_1) + A^{\#}(u_2).$$

similarly, $A^{\#}(cu_1) = cA^{\#}(u_1)$.

This map is also unique because if $B^{\#}(u) = A(v)$, then

$B^{\#}(v) = A^{\#}(v)$ for any $v \in V$. So $A^{\#} = B^{\#}$.

The dual space of a vector space

Def. Write V^* for the set of all linear functions $\ell : V \to \mathbb{R}$.

Define a vector space structure on V^* as follows: if $\ell_1, \ell_2 \in V^*$, then define the sum $\ell_1 + \ell_2$ to be the map $V \to \mathbb{R}$ by

$$(\ell_1 + \ell_2)(v) := \ell_1(v) + \ell_2(v).$$

If $\lambda \in \mathbb{R}$, then

$$(\lambda \ell)(v) := \lambda \ell(v).$$

V^* is called the dual space of V.

- Let $e_1, \dots, e_n$ be a basis of V. Then every vector $v \in V$ can be written uniquely as a sum $v = c_1 e_1 + \dots + c_n e_n$, $c_i \in \mathbb{R}$. Let $e_i^*(v) = c_i$, $e_i^*(v)$ is a linear function of V and hence $e_i^* \in V^*$.
- If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of V^*.

proof. Pick an arbitrary $\ell \in V^*$, we show it can be written by a linear combination of e_i^* s, i.e., $\ell = \sum_i \lambda_i e_i^*$ with $\lambda_i = \ell(e_i)$. To see this, note $\ell(e_j) = \sum_i \lambda_i e_i^*(e_j) = \lambda_j$. To show independence, if $\ell = 0$, then $\sum_i \lambda_i e_i^*(e_j) = 0$ for all j. So $\lambda_i = 0$. $\square$

- Let $A : V \to W$ a linear map. Given $\ell \in W^*$ is linear, $\ell \circ A$ is linear, and is an element of V^*. Denote $A^*\ell := \ell(A)$, and

$$\begin{aligned} A^* : W^* &\longrightarrow V^* \\ \ell &\mapsto A^*\ell \end{aligned}$$

We see A^* is linear:

$$A^*(\ell_1 + c\ell_2) = (\ell_1 + c\ell_2)(A)$$

- Let $A : V \to W$ a linear map. We call $A^* : W^* \to V^*$ define above the transpose of the map A.

A matrix description of A^*

- Let $e_1, \dots, e_n$ be a basis of V, $e_1^*, \dots, e_n^*$ be a basis of V^*
- Let $f_1, \dots, f_m$ a basis of W, $f_1^*, \dots, f_m^*$ a basis of W^*.
- Suppose $A e_j = \sum_{i=1}^{m} a_{i,j} f_i$ (*)

$$n \left\{ \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nm} \end{bmatrix}}_{m} \right. \overset{e_j}{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}} \begin{array}{l} \\ \\ \leftarrow j\text{th} \\ \\ \text{m column} \end{array} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \sum_{i=1}^{m} a_{ij} f_i$$

- Claim. The linear map A^* is defined, in terms of $f_1^*, \dots, f_m^*$ and $e_1^*, \dots, e_n^*$ by the transpose matrix $[a_{ji}]$.

proof. Let $A^* f_i^* = \sum_{j=1}^{n} c_{j,i} e_j^*$

$$m\left\{ \underbrace{\begin{bmatrix} & A^* & \end{bmatrix}}_{n} \right. \underbrace{f_i^*}_{\text{n rows}} = c_{1i} e_1^* + \cdots + c_{ni} \underbrace{e_j^*}_{\text{m rows}}$$

Then $A^* f_i^* (e_k) = c_{1i} e_1^* (e_k) + \cdots + c_{ni} e_j^* (e_k) = c_{ki}$

On the other hand, recall by definition $A^* \ell = \ell(A)$, so

$$A^* f_i^* (e_k) = f_i^* (A)(e_k) = f_i^* \sum_{j=1}^{m} a_{j,k} f_j \qquad \text{from (*)}$$

$$= \sum_{j=1}^{m} a_{j,k} f_i^* f_j = a_{i,k}$$

So $c_{k,i} = a_{i,k}$. ◻

1.3 Tensors

Def. Let V be an n-dim vector space and let V^k be the set of all k-tuples $(v_1, \dots, v_k)$, $v_i \in V$, i.e., the k-fold direct sum of V. A function

$$T : V^k \to \mathbb{R}$$

is said to be linear in its ith variable if, when we fix vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$, the map $V \to \mathbb{R}$ defined by

$$v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$$

is linear.

- If T is linear in its ith variable for $i = 1, \dots, k$ it is said to be k-linear, or alternatively a k-tensor. We write $\mathcal{L}^k(V)$ for the set of all k-tensors in V. We let 0-tensors to be real numbers.
- $\mathcal{L}^k(V)$ is a vector space. In particular, k-linear means linear when $k=1$, i.e., $\mathcal{L}^1(V) = V^*$.
- Let n, k be positive integers. A multi-index of n of length k is a k-tuple $I = (i_1, \dots, i_k)$ of integers with $1 \le i_r \le n$ for $r = 1, \dots, k$.
- Eg. There are n^2 $\to k$ of multi-indices of n of length 2 $\to k$.
- Fix a basis $e_1, \dots, e_n$ of V. For $T \in \mathcal{L}^k(V)$, write

$$T_I := T(e_{i_1}, \dots, e_{i_k})$$

for every multi-index I of length k.

- Prop. The real numbers T_I determine T, i.e., if T and T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.

proof. By induction on k. For $k=1$, can be proved using linearity.

Assume true for $k-1$, we now show true for k. For each e_i,

define a $(k-1)$-tensor $T_i : (v_1, \dots, v_{k-1}) \mapsto T(v_1, \dots, v_{k-1}, e_i)$.

Then for $v = c_1 e_1 + \cdots + c_n e_n$,

$$T(v_1, \dots, v_{k-1}, v) = \sum_{i=1}^{n} c_i T(v_1, \dots, v_{k-1}, e_i)$$

$$= \sum_{i=1}^{n} c_i T_i(v_1, \dots, v_{k-1}).$$

i.e., T_i determines T.

The tensor product operation

- If T_1 is a k-tensor, T_2 is an ℓ-tensor, one can define a $(k+\ell)$-tensor $T_1 \otimes T_2$ by setting:

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k)\, T_2(v_{k+1}, \dots, v_{k+\ell}).$$

This tensor is called the tensor product of T_1 and T_2.

- For 0-tensor, it is just scalar multiplication.

- Properties. For a k_1-tensor T_1, k_2-tensor T_2, k_3-tensor T_3 on V:

(1) Associativity. $(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3)$

(2) Distributivity. $\lambda (T_1 \otimes T_2) = (\lambda T_1) \otimes T_2 = T_1 \otimes (\lambda T_2)$

(3) Distributive laws. $(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$, if $k_1 = k_2$

$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$, if $k_2 = k_3$

- For $i = 1, \dots, k$, let $\ell_i \in V^*$, $T = \ell_1 \otimes \cdots \otimes \ell_k$. Thus,

$T(v_1, \dots, v_k) = \ell_1(v_1) \cdots \ell_k(v_k)$, called a decomposable k-tensor.

- Let $e_1, \dots, e_n$ be a basis of V, $e_1^*, \dots, e_n^*$ be a basis of V^*.

  Let $e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$. Then if J is another multi-index of length k, we have

$$e_I^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}$$

- Theorem. Let V be a vector space with basis $e_1, \dots, e_n$ and $0 \le k \le n$ be an integer. The k-tensors e_I^* are a basis of $\mathcal{L}^k(V)$.

proof. Given $T \in \mathcal{L}^k(V)$, let

$$T' = \sum_I T_I e_I^*, \qquad T_I := T(e_{i_1}, \dots, e_{i_k}).$$

Then $T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J$.

Previously, we proved T_Js determine T, so $T' = T$.

This shows e_I^*s are a spanning set of vectors for $\mathcal{L}^k(V)$.

To prove they are linearly independent, suppose

$$T' = \sum_I C_I e_I^* = 0 \Rightarrow \sum_I C_I e_I^*(e_{i_1}, \dots, e_{i_k}) = 0 \Rightarrow C_J = 0$$

Then with $T' = 0$, $C_J = 0$, for all J. □

- Recall there are exactly n^k multi-indices of length k and hence n^k basis vectors in the set $\{e_I^*\}_I$, so we proved

- Corrollary. Let V be an n-dimensional vector space. Then $\dim(\mathcal{L}^k(V)) = n^k$.

The pullback operation

Def. Let V and W be finite dimensional vector spaces and let $A: V \to W$ be a linear mapping. If $T \in \mathcal{I}^k(W)$, we define

$$A^* T : V^k \to \mathbb{R}$$

to be the function $(v_1, \dots, v_k) \mapsto T(Av_1, \dots, Av_k)$.

- Follows from the linearity of A, this function is linear in its i^{th} variable for all i, and hence a k-tensor. We call A^*T the pullback of T by the map A.

Proposition. The map

$$A^* : \mathcal{I}^k(W) \to \mathcal{I}^k(V)$$
$$T \mapsto A^* T$$

is a linear mapping.

proof. $A^*(T_1 + cT_2) = (T_1 + cT_2)(Av_1, \dots, Av_k)$

$= T_1(Av_1, \dots, Av_k) + cT_2(Av_1, \dots, Av_k)$ □

Proposition. For $T_1 \in \mathcal{I}^k(W)$, $T_2 \in \mathcal{I}^m(W)$, we have

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

proof. $A^*(T_1 \otimes T_2)(v_1, \dots, v_{k+m}) = (T_1 \otimes T_2)(Av_1, \dots, Av_{k+m})$

$= T_1(Av_1, \dots, Av_k)\, T_2(Av_{k+1}, \dots, Av_{k+m}) = A^*(T_1) \otimes A^*(T_2)$ □

Proposition. If U is a vector space, $B: U \to V$ a linear mapping, then

$(AB)^* T = B^* A^* T$.

proof. $(AB)^* T(v_1, \dots, v_k) = T((AB)v_1, \dots, (AB)v_k)$

$= A^* T(Bv_1, \dots, Bv_k) = B^* A^* T(v_1, \dots, v_k)$ □

1.4 Alternating k-tensors

Permutations

Def. • Let Σ_k be the k-element set $\Sigma_k := \{1, \dots, k\}$. A permutation of order k is a bijection $\sigma : \Sigma_k \xrightarrow{\sim} \Sigma_k$.

• Given two permutations σ_1 and σ_2, their product $\sigma_1 \sigma_2$ is the composition of $\sigma_1 \circ \sigma_2$: $i \mapsto \sigma_1(\sigma_2(i))$.

• Inverse permutation : $\sigma(i) = j \iff \sigma^{-1}(j) = i$

• Let S_k be the set of all permutations of order k, called the permutation group of Σ_k, or the symmetric group on k letters.

Lemma. The group S_k has $k!$ elements.

proof. For every $1 \le i < j \le k$ and $1 \le \ell \le k$, define the transposition

$$\tau_{i,j}(\ell) := \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i, j \end{cases}$$

If $j = i+1$, $\tau_{i,j}$ is called an elementary transposition.

- Base case: $k = 2$, $\sigma \in S_2$, $\sigma(k) = i$

  Then $\tau_{i,k}\sigma(k) = \tau_{1,2}(1) = 2$ for $i = 1$, i.e., $\sigma(k)$ maps k to i and $\tau_{i,k}$ switches it back. So $\tau_{i,k}\sigma$ is Σ_{k-1}.

- Inductive: For any k, $\tau_{i,k}\sigma(k)$ is Σ_{k-1}, assume has $(k-1)!$ permutations. Then $\tau_{i,k}$ has k choices on i, so total $k!$ permutations. □

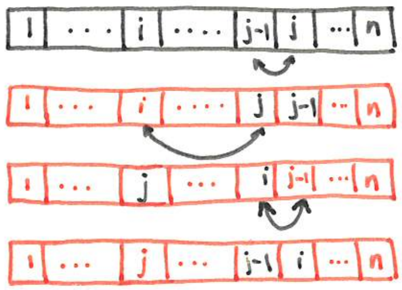
Theorem. Every permutation in S_k can be written as a product of (a finite number of) transpositions.

proof. Base case. $k = 2$, $S_k = \{id, \sigma(1) = 2, \sigma(2) = 1\} = \{\tau_{1,2}^2, \tau_{1,2}\}$

Inductive. Suppose true for $k-1$. Given $\sigma \in S_k$, then $\sigma(k) = i$ implies $\tau_{i,k}\sigma(k) = k$ and vice versa. So $\tau_{i,k}\sigma$ is a permutation of Σ_{k-1} and can be written as transposition. But any $\sigma \in S_k$ has $\sigma = \tau_{i,k}(\tau_{i,k}\sigma)$ thus is a product of transpositions. □

Theorem. Every transposition can be written as a product of elementary transpositions

proof. $\tau_{ij} = \tau_{j-1,j}\,\tau_{i,j-1}\,\tau_{j-1,j}$



Base case: $\tau_{i,i+1}$ is an elementary transposition.

Inductive: Suppose $\tau_{i,j-1}$ is a product of elementary transposition

Then the breakdown of τ_{ij} above concludes the hypothesis.

Corollary. Every permutation can be written as a product of elementary transpositions.

The sign of a permutation

Def. Let $x_1, \dots, x_k$ be the coordinate functions on $\mathbb{R}^k$. For $\sigma \in S^k$, define

$$(-1)^\sigma := \prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}, \quad \text{the sign of } \sigma.$$

Example: $\sigma = (1\ 2)$, x_1, x_2, x_3:

$$(-1)^\sigma := \frac{x_2 - x_1}{x_1 - x_2} \cdot \frac{\cancel{x_2 - x_3}}{\cancel{x_1 - x_3}} \cdot \frac{\cancel{x_1 - x_3}}{\cancel{x_2 - x_3}} = -1$$

$$\sigma = (1\ 2\ 3) = (1\ 3)(1\ 2), \quad (-1)^\sigma := \frac{\cancel{x_2 - x_3}}{x_1 - x_2} \cdot \frac{x_2 - x_1}{x_1 - x_3} \cdot \frac{x_3 - x_1}{\cancel{x_2 - x_3}} = 1$$

Claim. For $\sigma, \tau \in S_k$, $(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau$. That is, the sign defines a homomorphism $S_k \to \{\pm 1\}$.

proof.

$$(-1)^{\sigma\tau} := \prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_i - x_j} = \prod_{i<j} \frac{x_{\sigma\tau(i)} - x_{\sigma\tau(j)}}{x_{\tau(i)} - x_{\tau(j)}} \cdot \frac{x_{\tau(i)} - x_{\tau(j)}}{x_i - x_j}$$

$$= \prod_{p<q} \frac{x_{\sigma(p)} - x_{\sigma(q)}}{x_p - x_q} \cdot (-1)^\tau \qquad \begin{cases} \text{if } \tau(i) < \tau(j),\ p = \tau(i),\ q = \tau(j) \\ \text{otherwise, } p = \tau(j),\ q = \tau(i) \end{cases}$$

$$= (-1)^\sigma (-1)^\tau.$$

Proposition. If σ is the product of an odd number of transpositions, $(-1)^\sigma = -1$, otherwise, $(-1)^\sigma = 1$.

Alternation

Def. Let V be an n-dim vector space and $T \in \mathcal{I}^k(V)$ a k-tensor.

For $\sigma \in S_k$, define $T^\sigma \in \mathcal{I}^k(V)$ to be the k-tensor

$$T^\sigma(v_1, \dots, v_k) := T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}).$$

Proposition. (1) If $T = \ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in V^*$, then

$$T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}.$$

proof. $T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$

$$= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)})$$

Set $\sigma^{-1}(i) = q_i$, then $T^\sigma = \ell_{\sigma(q_1)}(v_{q_1}) \cdots \ell_{\sigma(q_k)}(v_{q_k})$ $\square$

(2) The assignment $T \to T^\sigma$ is a linear map $\mathcal{I}^k(V) \to \mathcal{I}^k(V)$.

proof. $(T_1 + cT_2)^\sigma := (T_1 + cT_2)(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$

$$= T_1(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) + cT_2(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}). \quad \square$$

(3) If $\sigma, \tau \in S_k$, $T^{\sigma\tau} = (T^\sigma)^\tau$.

proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$, $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k$

$$(T^\sigma)^\tau = \ell'_{\tau(1)} \otimes \cdots \otimes \ell'_{\tau(k)} = \ell_{\sigma(\tau(1))} \otimes \cdots \otimes \ell_{\sigma(\tau(k))} = T^{\sigma\tau}.$$

since $\ell'_{\tau(j)} = \ell_{\sigma(\tau(j))}$.

Example. $T = \ell_1 \otimes \ell_2 \otimes \ell_3 \otimes \ell_4$, $\sigma = (1\ 2\ 3)$, $\tau = (2\ 3\ 4)$

$T^\sigma = \ell_2 \otimes \ell_3 \otimes \ell_1 \otimes \ell_4$ $\quad T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(4)}) = \ell_1(v_3)\ell_2(v_1)\ell_3(v_2)\ell_4(v_4)$

$\tau^{-1}(3)$ $\tau^{-1}(1)$ $\tau^{-1}(2)$ $\tau^{-1}(4)$

$(T^\sigma)^\tau = \ell_2 \otimes \ell_1 \otimes \ell_4 \otimes \ell_3$ $\quad T(v_{(\sigma\tau)^{-1}(1)}, \dots, v_{(\sigma\tau)^{-1}(4)}) = \ell_1(v_2)\ell_2(v_1)\ell_3(v_4)\ell_4(v_3)$

$\sigma\tau = (1\ 2\ 3)(2\ 3\ 4) = (1\ 2)(3\ 4)$

$\tau\sigma = (2\ 3\ 4)(1\ 2\ 3) = (1\ 3)(2\ 4)$

$T^{\sigma\tau} = \ell_2 \otimes \ell_1 \otimes \ell_4 \otimes \ell_3$

Def. Let V be a vector space and $k \geq 0$ an integer. A k-tensor $T \in \mathcal{I}^k(V)$ is alternating if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.

Def. Let V be a vector space and k a nonnegative integer. The alternation operation on $\mathcal{L}^k(V)$ is defined as: given $T \in \mathcal{L}^k(V)$,

$$\mathrm{Alt}(T) := \sum_{\tau \in S_k} (-1)^{\tau} T^{\tau}.$$

Proposition. For $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$,

(1) $\mathrm{Alt}(T)^{\sigma} = (-1)^{\sigma} \mathrm{Alt}\, T$ $\qquad$ $\mathrm{Alt}(T)^{\sigma} = (\mathrm{Alt}\, T)^{\sigma}$

proof. $\mathrm{Alt}(T)^{\sigma} = \left(\sum_{\tau \in S_k} (-1)^{\tau} T^{\tau}\right)^{\sigma} = (-1)^{\sigma}\left(\sum_{\tau \in S_k} (-1)^{\tau}(-1)^{\sigma} T^{\tau\sigma}\right)$

$$= (-1)^{\sigma}\left(\sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma}\right) = (-1)^{\sigma} \mathrm{Alt}\, T$$

(2) If $T \in A^k(V)$, then $\mathrm{Alt}\, T = k!\, T$.

proof. If $T \in A^k(V)$,

$$\mathrm{Alt}\, T = \sum_{\tau \in S_k} (-1)^{\tau} T^{\tau} = \sum_{\tau \in S_k} (-1)^{\tau}(-1)^{\tau} T = k!\, T.$$

(3) $(\mathrm{Alt}\, T)^{\sigma} = \mathrm{Alt}(T^{\sigma})$

proof. $\mathrm{Alt}(T^{\sigma}) = \sum_{\tau \in S^k} (-1)^{\tau} (T^{\sigma})^{\tau} = \sum_{\tau \in S^k} (-1)^{\sigma}(-1)^{\sigma\tau} T^{\sigma\tau}$

$$= (-1)^{\sigma} \sum_{\sigma\tau \in S^k} (-1)^{\sigma\tau} T^{\sigma\tau} = (-1)^{\sigma} \mathrm{Alt}(T) = (\mathrm{Alt}\, T)^{\sigma} \text{ (by 1)}$$

(4) The map $\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is linear.

$$T \mapsto \mathrm{Alt}\, T$$

proof. $\mathrm{Alt}\, T = \sum_{\tau \in S_k} (-1)^{\tau} T^{\tau}$, $T \mapsto T^{\sigma}$ is linear so $\mathrm{Alt}\, T$ is linear.

(This also concludes $A^k(V)$ being a linear subspace of $\mathcal{L}^k(V)$.)

Def. Let $I = (i_1, \dots, i_k)$ be a multi-index of length k.

(1) I is repeating if $i_r = i_s$ for some $r \neq s$.

(2) I is strictly increasing if $i_1 < i_2 < \cdots < i_r$.

(3) For $\sigma \in S_k$, write $I^{\sigma} := (i_{\sigma(1)}, \dots, i_{\sigma(k)})$.

Remark. If I is non-repeating, there is a unique $\sigma \in S_k$ so that I^{σ} is strictly increasing.

- Let $e_1, \dots, e_n$ be a basis of V, and let

$$e_I^* := e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*, \quad \psi_I := \mathrm{Alt}(e_I^*).$$

Prop. (1) $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

proof. $(e_I^*)^\sigma = e_{i_{\sigma(1)}}^* \otimes \cdots \otimes e_{i_{\sigma(k)}}^* = e_{I^\sigma}^*$.

$\psi_{I^\sigma} := \mathrm{Alt}(e_{I^\sigma}^*) = \mathrm{Alt}(e_I^*)^\sigma = (-1)^\sigma \mathrm{Alt}(e_I^*)$, by previous (1)(3).

(2) If I is repeating, $\psi_I = 0$.

proof. Suppose $I = (i_1, \dots, i_k)$ with $i_r = i_s$ for $r \neq s$. Then if

$$\tau := \tau_{i_r, i_s},$$

then $e_I^* = e_{I^\tau}^* \Rightarrow \psi_I = \psi_{I^\tau} = (-1)^\tau \psi_I = -\psi_I \Rightarrow \psi_I = 0$. $\square$

(3) If I, J are strictly increasing,

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

proof. $\psi_I(e_{j_1}, \dots, e_{j_k}) := \mathrm{Alt}(e_I^*) := \sum_{\tau \in S_k} (-1)^\tau (e_I^*)^\tau = \sum_{\tau \in S_k} (-1)^\tau (e_{I^\tau}^*)$

$\left(\sum_\tau (-1)^\tau (e_{I^\tau}^*)\right)(e_{j_1}, \dots, e_{j_k}) = 1$ if $I^\tau = J$ for some τ.

But J is strictly increasing, so I^τ must be strictly increasing.

Therefore, τ must be the identity and $\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$ $\square$

- Let T be in $A^k(V)$. e_I^* are a basis of $\mathcal{L}^k(V)$, so

$$T = \sum_J a_J e_J^*, \quad a_J \in \mathbb{R}.$$

$T \in A^k(V) \Rightarrow k!\, T = \mathrm{Alt}(T)$. Thus,

$$T = \frac{1}{k!} \mathrm{Alt}\left(\sum_J a_J e_J^*\right) = \frac{1}{k!} \sum_J a_J \mathrm{Alt}(e_J^*) = \sum b_J \psi_J.$$

For every non-repeating term, J, we have $J = I^\sigma$ for some σ, where I is strictly increasing. So $\psi_J := \mathrm{Alt}(e_J^*) = \mathrm{Alt}(e_{I^\sigma}^*) = \mathrm{Alt}(e_I^*)^\sigma = (-1)^\sigma \psi_I$

- Thus, we can write T as a sum $T = \sum_I c_I \psi_I$, with I strictly increasing.

Claim. The C_I's are unique.

proof. For J strictly increasing, we have

$$T(e_{j_1}, \ldots, e_{j_k}) = \sum_I C_I \psi_I (e_{j_1}, \ldots, e_{j_k}) = C_J .$$

Therefore, if $\sum_I C_I \psi_I = 0$, $C_I = 0$. So ψ_I are linearly independent

Previously, we showed ψ_I is a spanning set for $\mathcal{A}^k(V)$, thus,

C_I's are unique.

And as a result:

Proposition. The alternating tensors ψ_I with I strictly increasing, are a basis for $\mathcal{A}^k(V)$.

- Thus, $\dim \mathcal{A}^k(V)$ is the number of strictly increasing multi-indices I of length k, which is $\binom{n}{k}$, thinking as taking a subset of k elements out of n elements.
- If $k > n$, for every multi-index $I = (i_1, \ldots, i_k)$ we have $\psi_I = 0$ and $\mathcal{A}^k(V) = 0$.

1.5 The space $\Lambda^k(V^*)$

- The image of the alternation operation

$$\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$$

is $\mathcal{A}^k(V)$. We will compute the kernel of Alt.

- Def. A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_1, \dots, \ell_k \in V^*$, is redundant if for some index i we have $\ell_i = \ell_{i+1}$. Let $\mathcal{I}^k(V) \subset \mathcal{L}^k(V)$ be the linear span of the set of redundant k-tensors. Set $\mathcal{I}^1(V) := 0$.

Proposition. If $T \in \mathcal{I}^k(V)$, then $\mathrm{Alt}\, T = 0$.

proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Set $\tau = \tau_{i,i+1}$. Then $T^\tau = T$, $(-1)^\tau = -1$. So $\mathrm{Alt}\, T = \mathrm{Alt}(T^\tau) = (-1)^\tau \mathrm{Alt}\, T$. □

Proposition. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then $T \otimes T'$, and $T' \otimes T$ are in $\mathcal{I}^{r+s}(V)$.

proof. Because $T \in \mathcal{I}^r(V)$, by definition, T is decomposable and $T = \ell_1 \otimes \cdots \otimes \ell_r$, with $\ell_i = \ell_{i+1}$ for some i. Then $T \otimes T' = \ell_1 \otimes \cdots \otimes \ell_r \otimes T'$ is redundant and similarly for $T' \otimes T$. □

Proposition. If $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, then $T^\sigma = (-1)^\sigma T + S$, where S is in $\mathcal{I}^k(V)$.

proof. Note elements in $\mathcal{L}^k$ can be written as a linear combination of e_I^*, and therefore decomposable. Therefore, we can work with decomposable tensors $T = \ell_1 \otimes \cdots \otimes \ell_k$ and proceed the proof with induction.

Base case: $T = \ell_1 \otimes \ell_2$. $T^{\sigma} = (-1)^\sigma T + (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$ ($\sigma = \tau_{1,2}$)

Set $\sigma = \tau_{i,i+1}$. $T^\sigma = (-1)^\sigma T + T_1 \otimes (\ell_i \otimes \ell_{i+1} + \ell_{i+1} \otimes \ell_i) \otimes T_2$

$T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-1}$, $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$. This concludes the base case that $T^\sigma = (-1)^\sigma T + S$ for σ being $\tau_{i,i+1}$.

Inductive Step. Since σ can be written as m elementary transpositions, We assume it's true for $m-1$, and show the m case.

Let $\sigma = \beta\tau$, where β is a product of $m-1$ elementary transpositions, and τ is an elementary transposition. Then

$$T^\sigma = T^{\beta\tau} = (-1)^\tau (T^\beta) + S, \quad S \in \mathcal{I}^k(V)$$

$$= (-1)^\tau \left((-1)^\beta T + S' \right) + S, \quad S' \in \mathcal{I}^k(V), \text{ by hypothesis}$$

By claim 1.4.9, $(-1)^\tau (-1)^\beta = (-1)^{\tau\beta}$, so

$$T^\sigma = (-1)^\tau (-1)^\beta T + (-1)^\tau S' + S$$

$$= (-1)^\sigma T + S'', \quad S'' = (-1)^\tau S' + S \in \mathcal{I}^k(V). \quad \square$$

Corollary. If $T \in \mathcal{L}^k(V)$, then $\mathrm{Alt}(T) = k!\,T + W$, $W \in \mathcal{I}^k(V)$.

proof. $\mathrm{Alt}(T) := \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma = \sum_{\sigma \in S_k} (-1)^\sigma \left((-1)^\sigma T + W_\sigma \right)$

$$= \left(\sum_{\sigma \in S_k} T \right) + \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma = k!\,T + W, \quad W = \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma. \quad \square$$

Corollary. Let V be a vector space and $k \geq 1$. Then

$$\mathcal{I}^k(V) = \ker(\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V)).$$

proof. If $T \in \mathcal{I}^k(V)$, then $\mathrm{Alt}(T) = 0$ as previously proved

If $\mathrm{Alt}(T) = 0$, then $\mathrm{Alt}(T) = k!\,T + W = 0$ with $W \in \mathcal{I}^k(V)$. $T = \frac{1}{k!} W$ $\square$

Theorem. Every element $T \in \mathcal{L}^k(V)$ can be written uniquely as $T = T_1 + T_2$, where $T_1 \in \mathcal{A}^k(V)$, $T_2 \in \mathcal{I}^k(V)$.

proof. Given for any $T \in \mathcal{L}^k(V)$, we have $\mathrm{Alt}(T) = k!\,T + W$, so we have

$$T = \frac{1}{k!}\mathrm{Alt}(T) - \frac{1}{k!} W, \quad \text{where } T_1 = \frac{1}{k!}\mathrm{Alt}(T), \ T_2 = \frac{1}{k!} W.$$

To show $T = T_1 + T_2$ is unique, it is the same as to show if $T = T_1' + T_2'$, then $0 = \underbrace{(T_1 - T_1')}_{\tilde{T}_1} + \underbrace{(T_2 - T_2')}_{\tilde{T}_2}$ with $T_1 - T_1' = 0$ and $T_2 - T_2' = 0$. That is, if $\tilde{T}_1 + \tilde{T}_2 = 0$, then $\tilde{T}_1 = \tilde{T}_2 = 0$.

But if $\mathrm{Alt}(\tilde{T}_1 + \tilde{T}_2) = 0$ then $\frac{1}{k!}\mathrm{Alt}(T) = T_1$ implies $T_1 = 0$. $\square$

Def. Let V be a finite-dimensional vector space and $k \geq 0$. Define

$$\Lambda^k(V^*) := \mathcal{L}^k(V) / \mathcal{I}^k(V),$$

i.e., let $\Lambda^k(V^*)$ be the quotient of the vector space $\mathcal{L}^k(V)$ by the subspace $\mathcal{I}^k(V)$. Recall there is a linear map

$$\begin{aligned} \pi : V &\to V/W \\ v &\mapsto v + W \end{aligned}$$

that maps surjectively onto V/W. Thus, we have the linear map

$$\begin{aligned} \pi : \mathcal{L}^k(V) &\longrightarrow \Lambda^k(V^*) \\ T &\mapsto T + \mathcal{I}^k(V), \end{aligned}$$

which is onto and has $\mathcal{I}^k(V)$ as kernel.

Theorem. The map $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^*)$.

proof. Every $\mathcal{I}^k(V)$ coset $T + \mathcal{I}^k(V)$ contains a unique element T_1 of $\mathcal{A}^k(V)$. Hence for every element of $\Lambda^k(V^*)$, there is a unique element of $\mathcal{A}^k(V)$, that gets mapped onto $\Lambda^k(V^*)$ by π. $\square$

Remark. $\Lambda^k(V^*)$ and $\mathcal{A}^k(V)$ are isomorphic vector spaces.

1.6 The Wedge Product

Def. Given $\omega_i \in \Lambda^{k_i}(V^*)$, $i = 1, 2$. Since

$$\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$$
$$T \mapsto T + \mathcal{I}^k(V)$$

we can find a $T_i \in \mathcal{L}^{k_i}(V)$ with $\omega_i = \pi(T_i)$. Then

$$T_1 \otimes T_2 \in \mathcal{L}^{k_1+k_2}(V).$$

The wedge product $\omega_1 \wedge \omega_2$ is defined by

$$\omega_1 \wedge \omega_2 := \pi(T_1 \otimes T_2) \in \Lambda^{k_1+k_2}(V^*).$$

Claim. This wedge product is well defined, i.e., does not depend on choices of T_1 and T_2.

proof. Let $\pi(T_1) = \pi(T_1') = \omega_1$. Then $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^k(V)$.

Then $$T_1' \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2$$

But $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}(V)$, so

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2).$$

Similarly: $\pi(T_1 \otimes T_2') = \pi(T_1 \otimes T_2)$. □

More generally, let $\omega_i \in \Lambda^{k_i}(V^*)$ for $i = 1, 2, 3$, and let $\omega_i = \pi(T_i)$, $T_i \in \mathcal{L}^{k_i}(V)$. Define

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \in \Lambda^{k_1+k_2+k_3}(V^*)$$

by setting $\omega_1 \wedge \omega_2 \wedge \omega_3 = \pi(T_1 \otimes T_2 \otimes T_3)$. Similarly, we can show this is well-defined and

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3).$$

- $\lambda(\omega_1 \wedge \omega_2) = (\lambda \omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda \omega_2)$.

proof. $\lambda(\omega_1 \wedge \omega_2) = \lambda \pi(T_1 \otimes T_2) = \pi(\lambda T_1 \otimes T_2) = \pi(T_1 \otimes \lambda T_2)$. □

- $(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$, $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$

proof. $(\omega_1 + \omega_2) \wedge \omega_3 = \pi((T_1 + T_2) \otimes T_3) = \pi(T_1 \otimes T_3 + T_2 \otimes T_3)$.

Same for the right distributive law. □

- $\mathcal{I}^1(V) = 0$, there is no redundant 1-tensor. Thus, $\Lambda^1(V^*) = V^* = \mathcal{L}^1(V)$.

- Let $\ell_1, \dots, \ell_k \in V^* = \Lambda^1(V^*)$. If $T = \ell_1 \otimes \cdots \otimes \ell_k$, then
$$\ell_1 \wedge \cdots \wedge \ell_k = \pi(T) \in \Lambda^k(V^*),$$
called a decomposable element of $\Lambda^k(V^*)$.
- For $\sigma \in S_k$, $\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$.

proof. We have showed that for all $T \in \mathcal{L}^k(V)$,
$$T^\sigma = (-1)^\sigma T + W$$
for some $W \in \mathcal{I}^k(V)$. And π is linear, so
$$\pi(T^\sigma) = (-1)^\sigma \pi(T) + \pi(W) = (-1)^\sigma \pi(T).$$
If $T = \ell_1 \otimes \cdots \otimes \ell_k$, $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$, so
$$\pi(T^\sigma) = \ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k. \qquad \square$$

Theorem. If $\omega_1 \in \Lambda^r(V^*)$, $\omega_2 \in \Lambda^s(V^*)$, then
$$\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1.$$

proof. Suffice to prove for decomposable elements ω_1, since $\omega_1 := \pi(T_1)$ and T can be written as a linear combination of decomposible elements, and use the linearity of π. Thus
$$\omega_1 \wedge \omega_2 = (\ell_1 \wedge \cdots \wedge \ell_r) \wedge (\ell_1' \wedge \cdots \wedge \ell_s').$$
Since $\ell_1 \wedge \ell_2 = -\ell_2 \wedge \ell_1$, and moving $\ell_1', \dots, \ell_s'$ infront of $\ell_1, \dots, \ell_r$ involves rs operations. Thus, $\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1$. $\square$

- Let $e_1, \dots, e_n$ be a basis of V and $e_1^*, \dots, e_n^*$ be the dual basis of V^*. For every multi-index I of length k.
$$e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = \pi(e_I^*) = \pi(e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*)$$

Theorem. $e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = \pi(e_I^*)$ with I strictly increasing, are a basis of $\Lambda^k(V^*)$.

proof. The elements $\psi_I = \mathrm{Alt}(e_I^*)$ for I strictly increasing, are basis vectors for $\mathcal{A}^k(V)$. So $\pi(\psi_I)$ are basis vectors of $\Lambda^k(V^*)$ since by Theorem 1.5.13, $\pi: \mathcal{L}^k(V) \to \Lambda^k(V)$ maps $\mathcal{A}^k(V)$ bijectively to $\Lambda^k(V^*)$. This concludes $\pi(e_I^*)$ is a basis of $\Lambda^k(V^*)$ since
$$\pi(\psi_I) = \pi\left(\sum_\sigma (-1)^\sigma (e_I^*)^\sigma\right) = \sum_\sigma (-1)^\sigma \pi\left((-1)^\sigma (e_I^*)\right) = k!\,\pi(e_I^*). \qquad \square$$

1.7 The interior product

- This section describes another product operation on $\Lambda^k(V^*)$.

Def. Let V be a vector space and k a non-negative integer. Given $T \in \mathcal{L}^k(V)$ and $v \in V$, let $\iota_v T$ be the $(k-1)$-tensor which takes

$$(\iota_v T)(v_1, \dots, v_{k-1}) := \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots v_{k-1})$$

on the $(k-1)$-tuple of vectors.

- From the definition, it follows that if $v = v_1 + v_2$, $T = T_1 + T_2$:

$$\iota_{v_1} T + \iota_{v_2} T = \iota_v T, \quad \iota_v T = \iota_v T_1 + \iota_v T_2.$$

Lemma. If T is the decomposable k-tensor $l_1 \otimes \cdots \otimes l_k$, then

$$\iota_v T = \sum_{r=1}^{k} (-1)^{r-1} l_r(v)\, l_1 \otimes \cdots \otimes \hat{l}_r \otimes \cdots \otimes l_k.$$

proof.
$$\begin{aligned}
\iota_v T(v_1, \dots, v_{k-1}) &:= \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} (-1)^{r-1} l_1(v_1) \cdots l_{r-1}(v_{r-1})\, l_r(v) \cdots l_k(v_{k-1}) \\
&= \sum_{r=1}^{k} (-1)^{r-1} l_r(v) \left(l_1 \otimes \cdots \otimes \hat{l}_r \otimes l_{r+1} \otimes \cdots \otimes l_k \right). \quad \square
\end{aligned}$$

Lemma. If $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$, then

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2.$$

proof.
$$\begin{aligned}
\iota_v(T_1 \otimes T_2) &= \sum_{r=1}^{p+q} (-1)^{r-1} T_1 \otimes T_2 (v_1, \dots, v_{r-1}, v, v_r, \dots, v_{p+q-1}) \\
&= \sum_{r=1}^{p} (-1)^{r-1} T_1(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{p-1}) T_2(v_p, \dots, v_{p+q-1}) \\
&\quad + \sum_{r=p+1}^{q} (-1)^{r-1} T_1(v_1, \dots, v_p) T_2(v_{p+1}, \dots, v_{r-1}, v, v_r, \dots, v_{p+q-1}) \\
&= \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2. \quad \square
\end{aligned}$$

Lemma. Let V be a vector space and $T \in \mathcal{L}^k(V)$. Then for all $v \in V$:

$$\iota_v(\iota_v T) = 0.$$

proof. Suffice to prove this for decomposable tensors due to linearity.

Base case is trivially true since it is a summation of nothing.

Assume true for $(k-1)$-tensors. Then

$$\iota_v(\ell_1 \otimes \cdots \otimes \ell_k) = \iota_v(\ell_1 \otimes \cdots \otimes \ell_{k-1}) \otimes \ell_k + (-1)^{k-1} \ell_1 \otimes \cdots \otimes \ell_{k-1} \otimes \iota_v \ell_k$$

$$\iota_v(\iota_v(\ell_1 \otimes \cdots \otimes \ell_k)) = \iota_v(\iota_v(\ell_1 \otimes \cdots \otimes \ell_{k-1}) \otimes \ell_k)$$
$$+ (-1)^{k-1} \iota_v(\ell_1 \otimes \cdots \otimes \ell_{k-1} \otimes \iota_v \ell_k)$$

$$= \iota_v(\iota_v(\ell_1 \otimes \cdots \otimes \ell_{k-1})) \otimes \ell_k + (-1)^{k-2} \iota_v(\ell_1 \otimes \cdots \otimes \ell_{k-1}) \otimes \iota_v \ell_k$$
$$+ (-1)^{k-1}(\iota_v(\ell_1 \otimes \cdots \otimes \ell_{k-1})) \otimes \iota_v \ell_k + (-1)^{2(k-1)} (\ell_1 \otimes \cdots \otimes \ell_{k-1}) \otimes \iota_v(\iota_v \ell_k)$$

The second and third terms cancel out, so we are done. □

Lemma. For $v_1, v_2 \in V$, $\iota_{v_1}\iota_{v_2} = -\iota_{v_2}\iota_{v_1}$.

proof. Set $v = v_1 + v_2$. Then $\iota_v = \iota_{v_1} + \iota_{v_2}$.

$$\iota_v \iota_v T = 0 \iff (\iota_{v_1} + \iota_{v_2})(\iota_{v_1} + \iota_{v_2}) T = 0$$

$$\iff (\iota_{v_1}\iota_{v_1} + \iota_{v_2}\iota_{v_2} + \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1}) T = 0$$

$$\iff \iota_{v_1}\iota_{v_2} T = -\iota_{v_2}\iota_{v_1} T.$$ □

- Now we show how to define the operation ι_v on $\Lambda^k(V^*)$.

Lemma. If $T \in \mathcal{L}^k(V)$ is redundant, so is $\iota_v T$.

proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$, $\ell \in V^*$, $T_1 \in \mathcal{L}^p(V)$, $T_2 \in \mathcal{L}^q(V)$.

Then $\iota_v T = \iota_v(T_1 \otimes \ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \underbrace{\ell \otimes \ell}_{\text{drop (redundant)}} \otimes \iota_v T_2$

$$= \iota_v T_1 \otimes \underbrace{\ell \otimes \ell}_{\text{drop}} \otimes T_2 + (-1)^p T_1 \otimes \iota_v(\ell \otimes \ell) \otimes T_2$$

However, $\iota_v(\ell \otimes \ell) = \iota_v \ell \otimes \ell + (-1)^1 \ell \otimes \iota_v \ell$, and $\iota_v \ell = \ell(v)$

So $\iota_v(\ell \otimes \ell) = \ell(v)\ell - \ell\,\ell(v) = 0.$ □

- Let π be the projection

$$\pi : \mathcal{I}^k(V) \to \Lambda^k(V^*)$$
$$T \mapsto T + \mathcal{I}^k(V)$$

of $\mathcal{I}^k(V)$ onto $\Lambda^k(V^*)$. For $\omega = \pi(T) \in \Lambda^k(V^*)$, define

$$\iota_v \omega := \pi(\iota_v T).$$

To show the definition is well-defined, note if $\omega = \pi(T_1) = \pi(T_2)$, then $T_1 - T_2 \in \mathcal{I}^k(V)$. So is $\iota_v(T_1 - T_2)$ by the previous lemma. Therefore,

$$\pi(\iota_v T_1) = \pi(\iota_v T_2).$$

- So we can define ι_v as a linear map:

$$\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*),$$

called the interior product operation.

- For $v, v_1, v_2 \in V$, $\omega \in \Lambda^k(V^*)$, $\omega_1 \in \Lambda^p(V^*)$, $\omega_2 \in \Lambda^q(V^*)$,
  - $\iota_{v_1+v_2}\omega = \iota_{v_1}\omega + \iota_{v_2}\omega$
  - $\iota_v(\omega_1 \wedge \omega_2) = \iota_v \omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \iota_v \omega_2$
  - $\iota_v(\iota_v \omega) = 0$
  - $\iota_{v_1}\iota_{v_2}\omega = -\iota_{v_2}\iota_{v_1}\omega$

- If $\omega = \ell_1 \wedge \cdots \wedge \ell_k$ decomposable, we have

$$\iota_v \omega = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v)\, \ell_1 \wedge \cdots \wedge \hat{\ell}_r \wedge \cdots \wedge \ell_k.$$

- If $\omega_I = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$, $1 \le i_1 < \cdots < i_k \le n$, then

$$\iota_{e_j}\omega_I = 0 \quad \text{if } j \notin I \quad \text{since} \quad e_{i_r}^*(e_j) = 0 \quad \text{for } j = 1, \dots, k$$

$$\iota_{e_j}\omega_I = (-1)^{r-1}\omega_{I_r} \quad \text{if } j = i_r \quad \text{since} \quad e_{i_r}^*(e_j) = 1 \quad \text{when } j = i_r \in I.$$

$I_r = (i_1, \dots, \hat{i}_r, \dots, i_k)$.

1.8 The pullback operation on $\Lambda^k(V^*)$

- V, W : vector spaces

  $A : V \to W$, linear map

  Given a k-tensor $T \in \mathcal{L}^k(W)$, the pullback A^*T is the k-tensor:

$$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

  in $\mathcal{L}^k(V)$.

- We will show how to define a similar pullback operation on $\Lambda^k(V^*)$.

Lemma. If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.

proof. Suffice to show this when T is a redundant k-tensor. Then

$$T = \ell_1 \otimes \cdots \otimes \ell_k, \quad \ell_r \in W^*, \quad \ell_i = \ell_{i+1}.$$

We have

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2). \quad \text{by 1.3.19}$$

So A^*T is redundant since $A^*\ell_i = A^*\ell_{i+1}$. ▫

- Let $\omega \in \Lambda^k(W^*)$, $\omega = \pi(T)$, $T \in \mathcal{L}^k(W)$. Define

(1.8.2) $\quad A^*\omega := \pi(A^*T)$.

Claim. $A^*\omega$ is well-defined.

proof. If $\omega = \pi(T) = \pi(T')$. Then $T = T' + S$ for $S \in \mathcal{I}^k(W)$.

$$A^*T = A^*T' + A^*S \Rightarrow A^*\omega = \pi(A^*T) = \pi(A^*T').$$

Prop. The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ is linear, and

$$\omega \mapsto A^*\omega$$

(1) if $\omega_i \in \Lambda^{k_i}(W^*)$, $i = 1, 2$, then $A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$.

Linearity: $\omega_1 = \pi(T_1)$, $\omega_2 = \pi(T_2)$, $\omega_1 + \lambda\omega_2 = \pi(T_1) + \pi(T_2)\lambda = \pi(T_1 + \lambda T_2)$

$$A^*(\omega_1 + \lambda\omega_2) := \pi(A^*(T_1 + \lambda T_2)) = \pi(A^*(T_1) + \lambda A^*(T_2)).$$

$$A^*(\omega_1 \wedge \omega_2) \overset{(1.6.1)}{:=} A^*(\pi(T_1 \otimes T_2)) \overset{(1.8.2)}{=} \pi(A^*(T_1 \otimes T_2)) \overset{(1.3.19)}{=} \pi(A^*(T_1) \otimes A^*(T_2))$$

$$\overset{(1.6.1)}{=} A^*\omega_1 \wedge A^*\omega_2.$$ ▫

(2) if U is a vector space and $B : U \to V$ a linear map, then for $\omega \in \Lambda^k(W^*)$, $B^* A^* \omega = (AB)^* \omega$.

proof. $B^* A^* \omega := \pi(B^* A^* T) \overset{(1.3.20)}{=} \pi((AB)^* T) := (AB)^* \omega$. □

• Now we use the pullback operation to define determinant.

Def. Let V be an n-dim vector space. Then $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$.

Thus, if $A : V \to V$ is a linear mapping, the induced pullback mapping

$$A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$$

is just multiplication by a constant, denoted $\det A$ and call it the determinant of A. By definition

$$A^* \omega := \det A(\omega) \quad , \quad \omega \in \Lambda^n(V^*).$$

Prop. If A and B are linear mappings of V into V, then

$$\det(AB) = \det A \det B$$

proof. $\det(AB)(\omega) = (AB)^* \omega = B^*(A^* \omega) = \det B (A^* \omega)$

$$= \det B \det A \ \omega$$

Prop. Write $id_V : V \to V$ for the identity map. Then $\det(id_V) = 1$.

proof. Because id_V^* is the identity map on $\Lambda^n(V^*)$, $id_V^* \omega = \omega$. Then

$$id_V^* \omega := \det(id_V)\omega = \omega \Rightarrow \det(id_V) = 1. \quad □$$

Prop. If $A : V \to V$ is not surjective, then $\det A = 0$.

proof. Let W be the image of A. If A is not onto, then $\dim W < n$.

so $\Lambda^n(W^*) = 0$ (at the end of Sec 1.4). Set

$$i_W : W \to V, \ w \mapsto w \quad , \quad i_W^* : V^* \to W^*, \ \omega \mapsto \omega$$

Let $A = i_W B$. Then for any $\omega \in \Lambda^n(V^*)$,

B: A regarded as a mapping from V to W

$$A^* \omega = (i_W B)^* \omega = B^* i_W^* \omega.$$

But $i_W^* \omega \in \Lambda^n(W^*) = 0$. so $A^* \omega = 0$. Then $\det(A)\omega = 0$. □

Matrix formula for determinant using wedge product

- V, W: n-dim vector spaces with basis e_i and f_i

  V^*, W^*: dual spaces with basis e_i^* and f_i^*

  $A : V \to W$, $\quad A e_j = \sum_{i=1}^{n} a_{i,j} f_i$,

  A: function (linear) sending e_j to $\sum_{i=1}^{n} a_{i,j} f_i$

  e_i, f_i: arbitrary basis for V, W

  $a_{i,j}$: the ij-entry of a matrix M_A.

Recall from Section 1.2, where we defined:

$$A^* \ell = \ell \circ A.$$

Therefore for a linear functional f_i^*, we have $A^* f_i^* = f_i^* \circ A$. Thus,

$$A^* f_i^* (e_j) = f_i^* \circ A (e_j) = f_i^* \Big(\sum_{k=1}^{n} a_{k,j} f_k \Big) = a_{i,j}$$

This shows

$$A^* f_i^* = \sum_{k=1}^{n} c_{k,i} e_k^*, \quad \text{so} \quad A^* f_i^* (e_j) = c_{j,i}, \quad \text{i.e.,} \quad c_{j,i} = a_{i,j}.$$

Thus, $\quad A^* f_i^* = \sum_{k=1}^{n} a_{i,k} e_k^*$.

- Consider now $A^*(f_1^* \wedge \cdots \wedge f_n^*)$, we just showed that

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = (A^* f_1^*) \wedge \cdots \wedge (A^* f_n^*).$$

$$= \Big(\sum_{k_1=1}^{n} a_{1,k_1} e_{k_1}^* \Big) \wedge \Big(\sum_{k_2=1}^{n} a_{2,k_2} e^* \Big) \wedge \cdots \wedge \Big(\sum_{k_n=1}^{n} a_{n,k_n} e_{k_n}^* \Big)$$

$$= \sum_{1 \le k_1, \ldots, k_n \le n} a_{1,k_1} a_{2,k_2} \cdots a_{n,k_n} e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*.$$

If the multi-index $k_1, \ldots, k_n$ is repeating, then $e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*$ is zero.

If it is not repeating, then we can write

$$k_i = \sigma(i), \quad i = 1, \ldots, n.$$

So $A^*(f_1^* \wedge \cdots \wedge f_n^*) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} e_{\sigma(1)}^* \wedge \cdots \wedge e_{\sigma(n)}^*$

Since $e_{\sigma(1)}^* \wedge \cdots \wedge e_{\sigma(n)}^* = (e_1^* \wedge \cdots \wedge e_n^*)^\sigma = (-1)^\sigma e_1^* \wedge \cdots \wedge e_n^*$, we have

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} e_1^* \wedge \cdots \wedge e_n^*$$

- We have

$$\det [a_{i,j}] = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

which is the determinant of the matrix $[a_{i,j}]$. So

$$A^* (f_1^* \wedge \cdots \wedge f_n^*) = \det [a_{i,j}] \, e_1^* \wedge \cdots \wedge e_n^*.$$

- If $V = W$, $e_i = f_i$, then $\omega = e_1^* \wedge \cdots \wedge e_n^* = f_1^* \wedge \cdots \wedge f_n^*$. From

$$A^* \omega = \det(A) \, \omega$$

we can conclude that

$$\begin{aligned} A^* \omega &= A^* (f_1^* \wedge \cdots \wedge f_n^*) = \det [a_{i,j}] \, e_1^* \wedge \cdots \wedge e_n^* \\ &= \det [a_{i,j}] \, \omega = \det(A). \end{aligned}$$

- So we have $\det [a_{i,j}] = \det A$.

This is the determinant of the matrix $[a_{i,j}]$

this is the determinant of the linear operator A.

1.9 Orientation

Def. If $\ell \subset \mathbb{R}^2$ is a line through the origin, then $\ell \setminus \{0\}$ has two connected components and an orientation of ℓ is a choice of one of these components.

- If L is a one-dimensional vector space, then $L \setminus \{0\}$ consists of two components; namely, if v is an element of $L \setminus \{0\}$, then these two components are:

$$L_1 = \{\lambda v \mid \lambda > 0\}, \quad L_2 = \{\lambda v \mid \lambda < 0\}.$$

Def. Let L be a one-dimensional vector space. An orientation of L is a choice of one of a connected component of $L \setminus \{0\}$. The component chosen is denoted L_+, called the positive component of $L \setminus \{0\}$, and the other component L_-, called the negative component of $L \setminus \{0\}$.

Def. Let (L, L_+) be an oriented one-dimensional vector space. A vector $v \in L$ is positive oriented if $v \in L_+$.

Def. Let V be an n-dim vector space. An orientation of V is an orientation of the one-dimensional vector space $\Lambda^n(V^*)$.

- One important way of assigning an orientation to V is to choose a basis $e_1, \dots, e_n$ of V. Then if $e_1^*, \dots, e_n^*$ is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring $e_1^* \wedge \dots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$.

Def. Let V be an oriented n-dim vector space. We say an ordered basis $(e_1, \dots, e_n)$ of V is positively oriented if $e_1^* \wedge \dots \wedge e_n^*$ is in the positive component of $\Lambda^n(V^*)$.

- Suppose $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are bases of V and

$$e_j = \sum_{i=1}^{n} a_{i,j} f_i .$$

We just showed that (1.8.10) given $A e_j = \sum_{i=1}^{n} a_{i,j} f_i$,

$$A^*(f_1^* \wedge \cdots \wedge f_n^*) = \det[A_{i,j}]\, e_1^* \wedge \cdots \wedge e_n^*$$

Therefore, given $e_j = \sum_{i=1}^{n} a_{i,j} f_i$, we have

$$f_i^*(e_j) = f_i^*\Big(\sum_{k=1}^{n} a_{k,j} f_k\Big) = a_{i,j} .$$

Similar as before, we use this relation to discover coefficients of

$$f_i^* := \sum_{k=1}^{n} c_{k,i}\, e_k^* .$$

Using this formula, we have

$$f_i^*(e_j) = \sum_{k=1}^{n} c_{k,i}\, e_k^*(e_j) = c_{j,i}, \quad \text{i.e., } c_{j,i} = a_{i,j}$$

This shows $f_i^* := \sum_{k=1}^{n} a_{i,k} e_k^*$. Thus,

$$\begin{aligned} f_1^* \wedge \cdots \wedge f_n^* &= \Big(\sum_{k_1=1}^{n} a_{1,k_1} e_{k_1}^*\Big) \wedge \cdots \wedge \Big(\sum_{k_n=1}^{n} a_{n,k_n} e_{k_n}^*\Big) \\ &= \sum_{1 \le k_1, \dots, k_n \le n}^{n} a_{1,k_1} \cdots a_{n,k_n}\, e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* \\ &= \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\, e_1^* \wedge \cdots \wedge e_n^* \\ &= \det[a_{i,j}]\, e_1^* \wedge \cdots \wedge e_n^* . \end{aligned}$$

So we can conclude:

Prop. If $e_1, \dots, e_n$ is positively oriented, then $f_1, \dots, f_n$ is positively oriented if and only if $\det[a_{i,j}]$ is positive.

Cor. If $e_1, \dots, e_n$ is a positively oriented basis of V, then the basis

$$e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$$

is negatively oriented.

proof. Denote the new basis as $e_1', \dots, e_n'$ with $e_j' = e_j$ if $i \neq j$ and $e_j' = -e_j$ if $j = i$. Then $[a_{ij}] = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & -1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$, so $\det[a_{i,j}] < 0$

To get to discuss orientation on the quotient space, we first look at:

1.2.i. Let V be an n-dim vector space and W a k-dim vector space. Show that there exists a basis $e_1, \dots, e_n$ of V with the property that $e_1, \dots, e_k$ is a basis of W.

proof. Base case: Suppose $n-k=1$. Then $\{e_1, \dots, e_k\}$ is a basis of W, and $\{e_1, \dots, e_k, v\}$ is a basis of V for any $v \in V \setminus W$.

Inductive step: Suppose true for $n-k$, now show also true for $n-k+1$. Let $W' = \operatorname{span}\{e_1, \dots, e_{k-1}\}$, then $\{e_1, \dots, e_{k-1}, v'\}$ is a basis of W, for some $v' \in W \setminus W'$. By induction hypothesis, basis of W can be extended into a basis of V. $\square$

1.2.ii. Show that vectors $f_i := \pi(e_{k+i})$, $i = 1, \dots, n-k$ are a basis of V/W, where $\pi : V \to V/W$ is the quotient map.

proof. Linear independence: if $\sum_{i=1}^{n-k} c_i \pi(e_{k+i}) = 0 + W \Rightarrow \sum_{i=1}^{n-k} c_i e_{k+i} \in W \Rightarrow c_i = 0$

Span: $\forall v \in V$, $v = w + \sum_{i=1}^{n-k} b_i e_{k+i}$

$$\forall\ v + W \in V/W, \quad v = \left(\sum_{i=1}^{n-k} b_i e_{k+i}\right) + W = \left(\sum_{i=1}^{n-k} b_i \pi(e_{k+i})\right) W.$$

Theorem 1.9.9 Given orientations on V and V/W, one gets a natural orientation on W.

proof. Let $r = n-k$, π be the projection of V onto V/W. By 1.2.i and 1.2.ii we can choose a basis $e_1, \dots, e_n$ of V such that $e_{r+1}, \dots, e_n$ is a basis of W and $\pi(e_1), \dots, \pi(e_r)$ are a basis of V/W. By previous Corollary, we can ensure $\pi(e_1), \dots, \pi(e_r)$ is positively oriented by switching the sign of e_1 if needed, and $e_1, \dots, e_n$ positively oriented by switching e_n if needed. Now assign to W the orientation associated with basis $e_{r+1}, \dots, e_n$.

$$\overbrace{\underbrace{e_1, \dots, e_r}_{\substack{\pi(e_1), \dots, \pi(e_r) \\ V/W\ (+)}},\ \underbrace{e_{r+1}, \dots, e_n}_{W\ (+)}}^{V}$$

$e_{r+1}, \dots, e_n$ gives orientation of W.

Now we show this assignment is natural, i.e., it does not depend on the choice of basis $e_1, \dots, e_n$.

Let $f_1, \dots, f_n$ be another basis with the same properties.

Let $[a_{i,j}]$ be the matrix such that

$$e_j = \sum_{i=1}^{n} a_{i,j} f_i .$$

$[a_{i,j}]$ has the form $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B is the $r \times r$ matrix expressing the basis vectors $\pi(e_1), \dots, \pi(e_r)$ of V/W as a linear combination of $\pi(f_1), \dots, \pi(f_r)$:

$$\pi(e_1) = \pi\left(\sum_{i=1}^{r} a_{i,1} f_i\right) = \sum_{i=1}^{r} a_{i,1} \pi(f_i)$$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,r} & a_{1,r+1} & \cdots & a_{1,n} \\ \vdots & \begin{matrix} B \\ r\times r \end{matrix} & \vdots & \vdots & & \vdots \\ a_{r,1} & \cdots & a_{r,r} & a_{r,r+1} & \cdots & a_{r,n} \\ 0 & \cdots & 0 & a_{r+1,r+1} & \cdots & a_{r+1,n} \\ \vdots & & \vdots & \vdots & D = k\times k & \vdots \\ 0 & \cdots & 0 & a_{n,r+1} & \cdots & a_{n,n} \end{bmatrix}$$

And D is the $k \times k$ matrix representing the basis vector $e_{r+1}, \dots, e_n$ of W as a linear combination of $f_{r+1}, \dots, f_n$. Thus,

$$\det \overset{+}{A} = (\det \overset{+}{B})(\det D) \quad (+)$$

We just proved that if $\det D$ is positive, then $e_{r+1}, \dots, e_n$ is positively oriented implies $f_{r+1}, \dots, f_n$ is positively oriented. $\square$

Remark. Suppose $\dim W = n-1$. The choice of a vector $v \in V \setminus W$ gives one a basis vector $\pi(v)$ for the 1-dim space V/W. So if V is oriented, the choice of v gives a natural orientation on W.

Def. Let $A : V_1 \to V_2$ a bijective linear map of oriented n-dim vector space. We say A is orientation-preserving if for $\omega \in \Lambda^n(V_2^*)_+$, we have $A^*\omega$ is in $\Lambda^n(V_1^*)_+$.

Eg. If $V_1 = V_2$, then $A^*\omega = \det(A)\,\omega$. So A is orientation preserving if and only if $\det A > 0$.

Prop. Let V_1, V_2, and V_3 be oriented n-dim vector spaces and

$$A_i : V_i \xrightarrow{\sim} V_{i+1} \quad , \quad i=1,2$$

be bijective linear maps. Then if A_1 and A_2 are orientation preserving, so is $A_2 \circ A_1$.

proof.
$$A_1 : V_1 \longrightarrow V_2 \quad , \quad A_2 : V_2 \longrightarrow V_3$$
$$A_1^* : \Lambda^n(V_2^*) \to \Lambda^n(V_1^*), \; A_2^* : \Lambda^n(V_3^*) \longrightarrow \Lambda^n(V_2^*)$$

Now we look at $A_1^* A_2^* \omega$ for $\omega \in \Lambda^n(V_3^*)$:

$$A_1^* A_2^* \omega = (A_2 \circ A_1)^* \omega$$

But $A_2^* \omega$ is orientation preserving and $A_2^* \omega \in \Lambda^n(V_2^*)$, and $A_1^* A_2^* \omega = A_1^*(\omega')$, for $\omega' = A_2^* \omega \in \Lambda^n(V_2^*)$, which is also orientation preserving. So $A_2 \circ A_1$ is orientation preserving. $\square$