Chapter 3 Integration of Forms

- The change of variable formula asserts that if U and V are open subsets of $\mathbb{R}^n$ and $f: U \to V$ a C^1 diffeomorphism. Then for every continuous function $\phi: V \to \mathbb{R}$ the integral $\int_V \phi(y)\, dy$ exists if and only if the integral

$$\int_U (\phi \circ f)(x) \,|\det Df(x)|\, dx$$

exists. If these integrals exists, they are equal.

3.1 The Poincaré lemma for compactly supported forms on rectangles

- Def. Let v be a k-form on $\mathbb{R}^n$. We define the support of v by

$$\operatorname{supp}(v) := \overline{\{x \in \mathbb{R}^n \mid v_x \neq 0\}},$$

and we say that v is compactly supported if $\operatorname{supp}(v)$ is compact.

- We denote by $\Omega_c^k(\mathbb{R}^n)$ the set of all C^∞ k-forms which are compactly supported. If U is an open set of of $\mathbb{R}^n$, we denote by $\Omega_c^k(U)$ the set of all compactly supported k-forms whose support is contained in U.
- Let $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ be a compactly supported n-form with $f \in C_0^\infty(\mathbb{R}^n)$. We define the integral of ω over $\mathbb{R}^n$ to be the usual integral of f over $\mathbb{R}^n$: $\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f\, dx$. This integral is well-defined since the integration domain is bounded (follows from f is identically zero outside of some compact set).
- Now set Q to be the rectangle:

$$[a_1, b_1] \times \cdots \times [a_n, b_n].$$

Theorem 3.2.2 (Poincaré lemma for rectangles). Let ω be a compactly supported n-form with $\mathrm{supp}(\omega) \subset \mathrm{int}(Q)$. Then the following assertions are equivalent:

(1) $\int \omega = 0$

(2) There exists a compactly supported $(n-1)$-form μ with $\mathrm{supp}(\mu) \subset \mathrm{int}(Q)$ satisfying $d\mu = \omega$.

proof. (2) $\Rightarrow$ (1). Let $\mu = \sum_{i=1}^{n} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

Then $d\mu = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$

Note $\int \frac{\partial f_i}{\partial x_i} dx_i = f_i(x) \Big|_{(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n)}^{(x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n)} = 0$

since f_i is supported $\subset \mathrm{Int}\, Q$ (By Fubini, we first integrating with respect to x_i and then the rest.)

(1) $\Rightarrow$ (2). We replace Q to be U, an open subset of $\mathbb{R}^n$, and prove a stronger version (Theorem 3.2.3).

Theorem 3.2.3. We say U has property P if for every form $\omega \in \Omega_c^m(U)$ such that $\int_U \omega = 0$ we have $\omega \in d\Omega_c^{m-1}(U)$. Let $A \subset \mathbb{R}$ an open interval. If U has property P, $U \times A$ does as well.

proof. Let $(x, t) = (x_1, \ldots, x_{n-1}, t)$ be product coordinates on $U \times A$.

Given $\omega \in \Omega_c^n(U \times A)$, we can express ω as a wedge product $dt \wedge \alpha$,

$$\alpha = f(x,t)\, dx_1 \wedge \cdots \wedge dx_{n-1}$$

$$f \in C_0^\infty(U \times A).$$

Let $\theta \in \Omega_c^{n-1}(U)$ be the form

$$\theta = \left(\int_A f(x,t)\, dt \right) dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Then

$$\int_{\mathbb{R}^{n-1}} \theta = \int_{\mathbb{R}^n} f(x,t)\, dx\, dt = \int_{\mathbb{R}^n} \omega.$$

Given U has property P, $\theta = d\upsilon$ for some $\upsilon \in \Omega_c^{n-2}(U)$.

Let $\rho \in C^\infty(\mathbb{R})$ be a bump function which is supported on A with $\int_A \rho = 1$. Set

$$k = -\rho(t)\,dt \wedge \upsilon$$

Then by 2.4.3, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$, $\omega_1 \in \Omega^k(U)$,

$$dk = \rho(t)\,dt \wedge d\upsilon$$

So
$$\omega - dk = dt \wedge \alpha - \rho(t)\,dt \wedge \theta = dt \wedge (\alpha - \rho(t)\theta)$$
$$= dt \wedge \left(f(x,t) - \rho(t)\int_A f(x,t)\,dt\right) dx_1 \wedge \dots \wedge dx_{n-1}.$$

Note that set $u(x,t) = f(x,t) - \rho(t)\int_A f(x,t)\,dt$, then

$$\int u(x,t) = \int f(x,t)\,dt - \int \rho(t)\int_A f(x,t)\,dt$$
$$= \int f(x,t)\,dt - \int_A f(x,t)\,dt = 0.$$

Set $A = (a,b)$, let

$$v(x,t) = \int_a^t u(x,s)\,ds.$$

Then $v(x,a) = v(x,b) = 0$. Since $f \in C_0^\infty(U \times A)$, $\rho \in C^\infty(\mathbb{R})$,

$$v \in C_0^\infty(U \times A).$$

Meanwhile, $\frac{\partial}{\partial t} v = \frac{\partial}{\partial t}\int_a^t u(x,s)\,ds = u(t,s)$ by FTC.

Let
$$\gamma = v(x,t)\,dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Then
$$d\gamma = \frac{\partial v}{\partial t}(x,t)\,dt \wedge dx_1 \wedge \dots \wedge dx_{n-1}$$
$$= u(x,t)\,dt \wedge dx_1 \wedge \dots \wedge dx_{n-1} = \omega - dk$$

So
$$\omega = d(\gamma + k) \in d\Omega_c^{n-1}(U \times A)$$

since $\gamma \in \Omega_c^{n-1}(U \times A)$ and $k \in \Omega_c^{n-1}(U \times A)$.

Thus, $U \times A$ has property P. $\square$

3.2.i. Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class C^r, with support on the interval (a, b). Show the following are equivalent:

(1) $\int_a^b f(x)\,dx = 0$

(2) Exists $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) and $\frac{dg}{dx} = f$.

proof. (1) $\Rightarrow$ (2). Recall the Leibniz' Rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}\,dt.$$

Therefore, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)\frac{dx}{dx} - f(x)\frac{da}{dx} + \int_a^x \frac{\partial f(t)}{\partial x}\,dt = f(x)$.

So $g = \int_a^x f(t)\,dt$ is the desired function supported on (a, b) from (1)

(2) $\Rightarrow$ (1). Given $\frac{dg}{dx} = f$, $\int_a^b \frac{dg(x)}{dx} = \int_a^b f(x)\,dx$

So $g(b) - g(a) = \int_a^b f(x)\,dx$. Since g is supported on (a, b), we conclude $\int_a^b f(x)\,dx = 0$. $\square$

Therefore, combining 3.2.i and Theorem 3.2.3, we use induction on

$$\text{int } Q = A_1 \times \cdots \times A_n, \quad A_i = (a_i, b_i)$$

to conclude Q has property P. This proves (1) $\Rightarrow$ (2).

3.3 Poincaré lemma for compactly supported forms on open subsets of $\mathbb{R}^n$

The goal of this section is to prove Theorem 3.2.2 to arbitrary connected open subsets of $\mathbb{R}^n$. We start with some definitions and notations.

Definition. Let U be a connected open subsets of $\mathbb{R}^n$, and let ω_1 and ω_2 be compactly supported n-forms with support in U. We write $\omega_1 \sim \omega_2$ to denote the following statement: There exists a compactly supported $(n-1)$-form μ with support in U and $\omega_1 - \omega_2 = d\mu$.

- Fix a rectangle $Q_0 \subset U$ and an n-form ω_0 with $\operatorname{supp} \omega_0 \subset Q_0$ and $\int \omega_0 = 1$.
- Let $Q_i \subset U$, $i = 1, 2, 3, \ldots$ be a collection of rectangles with
$$U = \bigcup_{i=1}^{\infty} \operatorname{int}(Q).$$
Let ϕ_i be a partition of unity with $\operatorname{supp}(\phi_i) \subset \operatorname{int}(Q_i)$. Because U is compact, so it exists finite subcovers. Thus, we can write
$$\omega = \sum_{i=1}^{m} \phi_i \, \omega, \quad \omega \text{ compactly supported } n\text{-form},$$
and $\operatorname{supp}(\omega) \subset U$. Now we have $\operatorname{supp}(\phi_i \omega)$ contained in one of the open rectangles $\operatorname{int}(Q_i)$. Now we show we can join Q_0 to Q_i by a sequence of rectangles.

Lemma 3.3.3. There exists a sequence of rectangles $R_0, \ldots, R_{N+1}$ such that $R_0 = Q_0$, $R_{N+1} = Q_i$, and $\operatorname{int}(R_j) \cap \operatorname{int}(R_{j+1}) \neq \phi$.

proof. Set $A = \{ x \in U \mid \exists$ a sequence of rectangles R_j, $j = 1, \ldots, N+1$ with $R_0 = Q_0$, $x \in \operatorname{int} R_{N+1}$, $\operatorname{int}(R_j) \cap \operatorname{int}(R_{j+1}) \neq \phi. \}$

This set is open, because $\forall x' \in B_\varepsilon(x)$, $\exists R'_{N+1} \subset U$ such that $R_{N+1} \subset R'_{N+1}$. The complement is also open: take $y \in A^c$, and $y' \in B_\varepsilon(y)$, then $y' \in A^c$. Otherwise, assume $y' \in A$, because $y \in B_\varepsilon(y')$, $y \in A$. Since A is connected, $U = A$. □

Theorem 3.3.2. If ω is a compactly supported n-form with $\operatorname{supp}(\omega) \subset U$ and $c = \int \omega$, then $\omega \sim c\omega_0$.

proof. Suffice to show this holds for $\phi_i \omega$ on Q_i, because if for each i, we have $\operatorname{supp}(\phi_i \omega) \subset Q_i$, $C_i = \int \phi_i \omega$, we have

$$\phi_i \omega - C_i \omega_0 = d\mu_i,$$

then given $\omega = \sum \phi_i \omega$ and $c = \int \omega$,

$$\omega - c\omega_0 = \sum \phi_i \omega - \sum C_i \omega_0 = \sum d\mu_i = d\mu.$$

Therefore, to show this holds for $\phi_i \omega$, for each j, select a compactly supported n-form ν_j with $\operatorname{supp}(\nu_j) \subset \operatorname{int}(R_j) \cap \operatorname{int}(R_{j+1})$, and $\int \nu_j = 1$. Then $\nu_j - \nu_{j+1}$ is supported in R_{j+1} since

$$\operatorname{supp}(\nu_j) \subset \operatorname{int}(R_j) \cap \operatorname{int}(R_{j+1}),$$

$$\operatorname{supp}(\nu_{j+1}) \subset \operatorname{int}(R_{j+1}) \cap \operatorname{int}(R_{j+2}).$$

$\int_{R_{j+1}} \nu_j - \nu_{j+1} = 0$ since $\int \nu_j = \int \nu_{j+1} = 1$ both with support in R_{j+1}.

By Theorem 3.2.2, $\nu_j \sim \nu_{j+1}$.

Similarly, ω_0 has support in Q_0, $\int \omega_0 = 1$, same for ν_0. Thus,

$$\nu_0 \sim \omega_0.$$

Finally, $C_i = \int \phi_i \omega$, $\operatorname{supp}(\phi_i \omega) \subset Q_i$, $\operatorname{supp}(\nu_N) \subset \operatorname{int}(R_N) \cap \operatorname{int}(R_{N+1})$ so both $\phi_i \omega$ and ν_N are supported on Q_i since $R_{N+1} = Q_i$.

$\int \phi_i \omega = C_i$ and $\int \nu_N = 1$, thus, $\phi_i \omega \sim C_i \nu_N$. This proves

$$C_i \omega_0 \sim C_i \nu_0 \sim \cdots \sim C_i \nu_N \sim \omega. \qquad \square$$

Theorem 3.3.1 (Poincaré lemma for compactly supported forms). Let U be a connected open subset of $\mathbb{R}^n$ and let ω be a compactly supported n-form with $\operatorname{supp}(\omega) \subset U$. The following assertions are equivalent:

(1) $\int_{\mathbb{R}^n} \omega = 0$.

(2) There exists a compactly supported $(n-1)$-form μ with $\operatorname{supp} \mu \subset U$ and $\omega = d\mu$.

proof. (2) $\Rightarrow$ (1) We can use Theorem 3.2.2 since $\operatorname{supp}(\mu)$ can be contained in a large rectangle.

(1) $\Rightarrow$ (2) Give ω compactly supported n-form with $\operatorname{supp}(\omega) \subset U$ and $\int \omega = 0 = c$, we have $\omega - 0 = d\mu$ for some compactly supported $(n-1)$-form μ according to Theorem 3.3.2. $\square$

3.4 The degree of differentiable mapping

Def. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$. A continuous map $f: U \to V$ is proper if for every compact subset $K \subset V$, the preimage $f^{-1}(K)$ is compact in U.

- By 2.6.5, we know for 0-forms (functions), $f^*\phi$ is just the composition function $\phi \circ f \in C^\infty(U)$:

$$(f^*\phi)(p) = \phi(f(p)).$$

- Given a k-form $\omega \in \Omega^k(V)$ as a sum over multi-indices of length k,

$$\omega = \sum_I \phi_I \, dx_I,$$

the coefficients ϕ_I of dx_I are $C^\infty(V)$. Thus,

$$f^*\omega = f^*\left(\sum_I \phi_I \, dx_I\right).$$

By linearity of f^*, we have

$$f^*\omega = \sum_I f^*(\phi_I \, dx_I).$$

By 2.6.8, we have

$$f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2).$$

Thus, $\phi_I \, dx_I = \phi_I \wedge dx_I$, so

$$f^*\omega = \sum_I f^*(\phi_I \wedge dx_I) = \sum_I (f^*\phi_I \wedge f^* dx_I).$$

Using 2.6.5, $f^*\phi_I = \phi_I \circ f$. For simplicity,

$$f^* dx_I = f^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$
$$= (f^* dx_{i_1}) \wedge \cdots \wedge (f^* dx_{i_k})$$

By 2.6.7, $f^* d\phi = df^*\phi$, so $f^* dx_i = df^* x_i$. By 2.6.5, $df^* x_i = x_i \circ df = df_i$. Thus,

$$f^*\omega = \sum_I (f^*\phi_I \wedge f^* dx_I) = \sum_I ((\phi_I \circ f) \wedge df_I)$$
$$= \sum_I f^*\phi_I \, df_I.$$

- This formula concludes that if f is a C^∞ mapping and ω is a compactly supported k-form with support on V, $f^*\omega$ is a compactly supported k-form with support on U.

Goal: To show that if U and V are connected open subsets of $\mathbb{R}^n$ and $f: U \to V$ is a proper C^∞ mapping then there exists a topological invariant of f, which we will call its degree, denoted $\deg f$, such that the change of variables formula

$$\int_U f^*\omega = \deg f \int_V \omega$$

holds for all $\omega \in \Omega_c^n(V)$.

- 2.6.10 gives $f^*(dx_i) = df_i$. Let $\omega = dx_1 \wedge \cdots \wedge dx_n$, f maps from open subsets U of $\mathbb{R}^n$ to open subsets V of $\mathbb{R}^n$. Then

$$f^*\omega_p = (df_1)_p \wedge \cdots \wedge (df_n)_p$$

for all $p \in U$. Also,

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(p)(dx_j)_p$$

Thus, we have

$$f^*\omega_p = \left(\sum \frac{\partial f_1}{\partial x_j}(p)(dx_j)_p\right) \wedge \cdots \wedge \left(\sum \frac{\partial f_n}{\partial x_j}(p)(dx_j)_p\right)$$

In 1.8.10, we showed given $A^* f_j^* = \sum_{i=1}^n a_{j,i}\, e_i^*$, we have

$$(A^* f_j^*) \wedge \cdots \wedge (A^* f_n^*) = \sum_{1 \le k_1, \dots, k_n \le n} (a_{1,k_1} e_{k_1}^*) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^*).$$

$$= \sum_{1 \le k_1, \dots, k_n \le n} a_{1,k_1} \cdots a_{n,k_n}\, e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*$$

Replacing k_i by $\sigma(i)$

$$= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} (e_1^* \wedge \cdots \wedge e_n^*)$$

$$= \det[a_{i,j}]\, e_1^* \wedge \cdots \wedge e_n^*$$

Similar computation with $a_{i,k_i} = \sum_j \frac{\partial f_i}{\partial x_j}$, $e_{k_i}^* = dx_{k_i}$ gives

$$f^*\omega = \det\left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n.$$

- Therefore, given

$$\omega = \phi(y)\, dy_1 \wedge \cdots \wedge dy_n,$$

at $x \in U$, we have

$$f^*\omega = f^*(\phi(y) \wedge dy_1 \wedge \cdots \wedge dy_n)$$

$$= f^*(\phi(y_1)) \wedge f^*(dy_1 \wedge \cdots \wedge dy_n)$$

$$= (\phi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n.$$

Thus, $\int_U f^*\omega = \deg f \int_V \omega$ in coordinate form is

$$\int_U (\phi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \deg f \int_V \phi(y)\, dy_1 \wedge \cdots \wedge dy_n$$

Recall Theorem 3.3.1 (Poincaré Lemma for compactly supported forms). Let U be a connected open subset of $\mathbb{R}^n$ and let ω be a compactly supported n-form with $\operatorname{supp}(\omega) \subset U$. Then the following assertions are equivalent:

(1) $\int_{\mathbb{R}^n} \omega = 0$

(2) There exists a compactly supported $(n-1)$-form μ with $\operatorname{supp} \mu \subset U$ and $\omega = d\mu$.

Therefore, we can prove $\int_U f^*\omega = \deg f \int_V \omega$ for compactly supported n-form ω with $\operatorname{supp}(\omega) \subset V$. (We'll show why $\deg f$ is topological invariant in Sec 3.6.)

proof. Let ω_0 be a compactly supported n-form with $\operatorname{supp}(\omega_0) \subset V$, $\int \omega_0 = 1$. Set $\deg f := \int_U f^*\omega_0$. Let $c := \int_V \omega$, then

$$\int_{\mathbb{R}^n} (\omega - c\omega_0) = \int_{\mathbb{R}^n} \omega - \int_{\mathbb{R}^n} c\omega_0 = \int_V \omega - c\int_{\mathbb{R}^n} \omega_0 = 0.$$

By Theorem 3.3.1, there exists a compactly supported $(n-1)$ form μ such that $\omega - c\omega_0 = d\mu$. So $f^*\omega - cf^*\omega_0 = f^*(\omega - c\omega_0)$

Therefore, $f^*(\omega - c\omega_0) = f^* d\mu = df^*\mu$ by 2.6.7.

Using 3.3.1 again we have $\int_U f^*(\omega - c\omega_0) = 0$ so

$$\int_U f^*\omega = c\int_U f^*\omega_0 = \deg f \int_V \omega. \qquad \square$$

Proposition 3.4.4. Let U, V, W be connected open subsets of $\mathbb{R}^n$ and $f: U \to V$, $g: V \to W$ proper C^∞ maps, then

$$\deg(g \circ f) = \deg g \ \deg f$$

proof. Let ω be a compactly supported n-form with support on W. Then

$$(g \circ f)^*\omega = f^* g^* \omega$$

So

$$\int_U (g \circ f)^*\omega = \int_U f^* g^* \omega = \deg f \int_V g^*\omega$$

$$= \deg f \ \deg g \int_W \omega. \qquad \square$$

Theorem 3.4.6. Let A be a non-singular $n \times n$ matrix and $f_A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with A. Then $\deg(f_A) = +1$ if $\det A$ is positive and -1 if $\det A$ is negative.

The proof is done via Ex 3.4.v - 3.4.ix.

3.4.v. Let σ be a permutation of numbers $1, \dots, n$ and let

$$f_\sigma : \mathbb{R}^n \to \mathbb{R}^n \quad \text{diffeomorphism}$$

$$(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Prove that $\deg f_\sigma = (-1)^\sigma$.

proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^∞ function. Let

$$\phi(x) = \psi(x_1) \cdots \psi(x_n).$$

Let $\omega = \phi(x)\, dx_1 \wedge \cdots \wedge dx_n$, then

$$f_\sigma^*\omega = (\phi \circ f_\sigma)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n = (-1)^\sigma \omega$$

since $\phi \circ f_\sigma = \phi$ by commutativity of multiplicity, and $\det(Df(x))$ is $(-1)^\sigma$. So $\int_{\mathbb{R}^n} f_\sigma^*\omega = \deg f_\sigma \int_{\mathbb{R}^n} \omega \Rightarrow \deg f_\sigma = (-1)^\sigma$. $\square$

3.4. vi. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (x_1 + \lambda x_2, x_2, \dots, x_n).$$

Prove that $\deg f = 1$.

proof. Let $\omega = \phi(x_1, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$, $\phi: \mathbb{R}^n \to \mathbb{R}$ compactly supported and of class C^∞. Then

$$\int f^*\omega = \int (\phi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \int \phi(x_1 + \lambda x_2, x_2, \dots, x_n) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n$$

$$Df(x) = \begin{bmatrix} 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \Rightarrow \det(Df(x)) = 1.$$

$$\int f^*\omega = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \psi(x_1 + \lambda x_2)\, dx_1 \right) \psi(x_2) \cdots \psi(x_n)\, dx_2 \wedge \cdots \wedge dx_n$$

Note for $a \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \psi(t)\, dt = \int_{\mathbb{R}} \psi(t-a)\, dt,$$

i.e., translation on $\mathbb{R}$ does not affect the integration. So

$$\int f^*\omega = \int \phi(x)\, dx_1 \wedge \cdots \wedge dx_n = \int \omega = \deg f \int \omega \Rightarrow \deg f = 1. \quad \square$$

3.4. vii. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (\lambda x_1, x_2, \dots, x_n), \quad \lambda \neq 0.$$

Show $\deg f = 1$ if $\lambda > 0$, and $\deg f = -1$ if $\lambda < 0$.

proof. Set $\omega = \phi(x_1, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$. So

$$\int f^*\omega = \int (\phi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \lambda \int \psi(\lambda x_1) \psi(x_2) \cdots \psi(x_n)\, dx_1 \wedge \cdots \wedge dx_n$$

since $\det(Df(x)) = \begin{vmatrix} \lambda & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{vmatrix} = \lambda$. So

$|\lambda|$ to set off the change of sign when changing variable.

$$\int f^*\omega = \lambda \int \frac{1}{|\lambda|} \psi(x_1')\, d(x_1')\, \psi(x_2) \cdots \psi(x_n)\, dx_2 \wedge \cdots \wedge dx_n, \quad x_1' = \lambda x_1$$

So $\int f^*\omega = \int \omega = \pm \deg f \int \omega. \Rightarrow \deg f = 1$ if $\lambda > 0$, $\deg f = -1$ if $\lambda < 0$ $\square$

3.4.viii. (1) Let $e_1, \dots, e_n$ be the standard basis vectors of $\mathbb{R}^n$ and A, B, C are linear mappings defined by

$$Ae_i = \begin{cases} e_1 & i=1 \\ \sum_{j=2}^{n} a_{j,i}\, e_j & i \neq 1 \end{cases}$$

(3.4.10)
$$B e_i = \begin{cases} \sum_{j=1}^{n} b_j e_j & i=1 \\ e_i & i \neq 1 \end{cases}$$

$$Ce_i = \begin{cases} e_1 & i=1 \\ e_i + c_i e_1 & i \neq 1 \end{cases}$$

Show that
$$BACe_i = \begin{cases} \sum_{j=1}^{n} b_j e_j & i=1 \\ \left(\sum_{j=2}^{n} (a_{j,i} + c_i b_j)\, e_j\right) + c_i b_1 e_1, & i \neq 1 \end{cases}$$

for $i > 1$.

proof. For $i=1$, $BACe_i = BAe_1 = Be_1 = \sum_{j=1}^{n} b_j e_j$.

For $i \neq 1$, $BACe_i = BA(e_i + c_i e_1) = B\left(\sum_{j=2}^{n} a_{j,i} e_j + c_i e_1\right)$

$$= \sum_{j=2}^{n} a_{j,i} e_j + c_i \sum_{j=1}^{n} b_j e_j. \qquad \square$$

(2) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be the linear mapping

$$Le_i = \sum_{j=1}^{n} \ell_{j,i}\, e_j, \quad i = 1, \dots, n$$

Show that if $\ell_{1,1} \neq 0$, one can write L as a product $L = BAC$, where A, B, C are linear mappings of the form 3.4.10.

proof. $Le_1 = BACe_1 \Rightarrow \sum_{j=1}^{n} \ell_{j,1} e_j = \sum_{j=1}^{n} b_j e_j$.

Because e_j are basis, so $\ell_{j,1} = b_j$, for $j = 1, \dots, n$.

Now from $Le_i = BACe_i$ for $i \neq 1$, we have

$$\ell_{1,i} e_1 = c_i b_1 e_1, \quad \text{i.e.,} \quad \ell_{1,i} = c_i b_1$$

For $j \neq 1$, we have $\ell_{j,i} e_j = (a_{j,i} + c_i b_j) e_j$, i.e., $\ell_{j,i} = a_{j,i} + c_i b_j$. $\square$

(3) Suppose L is invertible. Conclude that A, B, C are invertible and verify that Theorem 3.4.6 holds for B and C.

proof. Recall for linear maps $A, B : \mathbb{R}^n \to \mathbb{R}^n$, we have

$$\dim(AB) \leq \min(\dim A, \dim B).$$

Given $L = BAC$ and L invertible, this concludes A, B, C are also invertible. Given

$$Be_i = \begin{cases} \sum_{j=1}^{n} b_j e_j & i = 1 \\ e_i & i \neq 1 \end{cases}$$

To find the degree of B and C, we use a sequence of transformations:

$$(e_1, e_2, \ldots, e_n) \mapsto (b_1 e_1, e_2, \ldots, e_n)$$

has degree $\operatorname{sgn}(b_1)$ by 3.4.vii.

$$(b_1 e_1, e_2, \ldots, e_n) \mapsto (b_1 e_1 + \sum_{j=2}^{n} b_j e_j, e_2, \ldots, e_n)$$

has degree 1 by a direct generalization of 3.4.vi. The matrix of

$$B = (e_1, e_2, \ldots, e_n) \mapsto (b_1 e_1 + \sum_{j=2}^{n} b_j e_j, e_2, \ldots, e_n)$$

is:

$$M_B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ b_2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ b_n & \cdots & \cdots & 1 \end{bmatrix} \to \det M_B = b_1.$$

Therefore, $\deg B = \operatorname{sgn}(b_1) = \operatorname{sgn}(\det M_B)$. Similarly, for

$$Ce_i = \begin{cases} e_1 & i = 1 \\ e_i + c_i e_1 & i \neq 1 \end{cases}$$

Following 3.4.vi, $\deg(C) = 1$, and

$$M_c = \begin{bmatrix} 1 & c_2 & \cdots & c_n \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \to \det M_C = 1.$$

$\square$

(4) Show inductively that Theorem 3.4.6 holds for A and use 3.4.5 to show it holds for L.

proof. We use induction to proceed.

Base case: $n=1$, $A_1 = [\ 1\ ]$, $f_A = id$. Set $\omega = \phi(x)\,dx$, $\int \omega = 1$.

$$\deg f_A = \int \phi \circ f_A(x)\,|\det Df_A|\,dx = 1.$$

$$A_1 = [\ -1\ ], \quad f_A = -id$$

Alternative: set $A_1 = [\lambda]$ and use 3.4.vii.

$$\deg f_A = \int \phi(-x)\,|-1|\,dx = \int \phi(-x) - d(-x) = -1$$

For induction hypothesis, we suppose Theorem 3.4.6 is true for A_{n-1}.

For inductive step, we set:

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \quad \text{and } A_n^* = \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Then we have:

$$f_{A_n}(x_1, \dots, x_n) = \left(x_1, \sum_{i=2}^n a_{2i} x_i, \dots, \sum_{i=2}^n a_{ni} x_i\right)$$

$$f_{A^*_{n-1}}(x_2, \dots, x_n) = \left(\sum_{i=2}^n a_{2i} x_i, \dots, \sum_{i=2}^n a_{ni} x_i\right)$$

Now set $\tilde{\omega} = \tilde{\phi}(x)\,dx$, $\tilde{\phi}(x) = \psi(x_1) \cdots \psi(x_n)$ with $\int \psi(x_1) = 1$.

$\hat{\omega} = \hat{\phi}(x)\,dx$, $\hat{\phi}(x) = \psi(x_2) \cdots \psi(x_n)$ with $\int \hat{\phi}(x) = 1$.

with $\psi: \mathbb{R} \to \mathbb{R}$ being a compactly supported C^∞ function

$$\deg f_{A^*_{n-1}} \int \hat{\omega} = \int \hat{\phi} \circ f_{A^*_{n-1}}(x)\,|\det Df_{A^*_{n-1}}(x)|\,dx_2 \wedge \cdots \wedge dx_n,$$

with $\int \hat{\omega} = \int \hat{\phi}(x)\,dx = \int \psi(x_1) \cdots \psi(x_{n-1})\,dx_2 \wedge \cdots \wedge dx_n = 1$.

$$\deg f_{A_n} \int \tilde{\omega} = \int \tilde{\phi} \circ f_{A_n}(x)\,|\det Df_{A_n}(x)|\,dx_1 \wedge \cdots \wedge dx_n,$$

with $\int \tilde{\omega} = \int \tilde{\phi}(x)\,dx = \int \psi(x_1) \cdots \psi(x_n)\,dx_1 \wedge \cdots \wedge dx_n$.

$$= \int_{\mathbb{R}^1_1} \psi(x_1)\,dx_1 \int_{\mathbb{R}^{n-1}_{2,\dots,n}} \psi(x_2) \cdots \psi(x_n)\,dx_1 \wedge \cdots \wedge dx_n \quad \text{by Fubini}$$

$$= \int_{\mathbb{R}^1_1} \psi(x_1)\,dx_1 = 1.$$

So $\deg f_A = \int \tilde{\Phi} \circ f_{A_n}(x) |\det Df_{A_n}(x)| \, dx_1 \wedge \cdots \wedge dx_n$

$$= \int \psi(x_1) \psi\left(\sum_{i=2}^{n} a_{2i} x_i\right) \cdots \psi\left(\sum_{i=2}^{n} a_{ni} x_i\right) |\det Df_{A_n}(x)| dx$$

$$= \int \psi(x_1) dx_1 |\det Df_A(x)| \int \psi\left(\sum_{i=2}^{n} a_{2i} x_i\right) \cdots \psi\left(\sum_{i=2}^{n} a_{ni} x_i\right) dx_2 \wedge \cdots$$

since $|\det Df_{A_n}(x)| = |\det Df_{A^*_{n-1}}(x)|$,

$$\deg f_A = \left(\int \psi(x_1) dx_1\right) \deg f_A = \deg f_A.$$

Now we have Theorem 3.4.6 holds for any matrix of the form A, B, and C, then for a generic matrix

$$L = BAC,$$

we have

$$\deg f_L = (\deg f_B)(\deg f_A)(\deg f_C)$$

by 3.4.5. Therefore, if $\det f_L > 0$, then

$$(\deg f_B)(\deg f_A)(\deg f_C) > 0$$

and $\deg f_L = 1$, similarly for $\det f_L < 0$. $\square$

Remark. Elementary matrix is a square matrix obtained by performing elementary row or column operation on an identity matrix, including interchanging two rows, multiply a row by a non-zero constant, and add a multiple of one row to another row. 3.4. iv - vii reveals the degree of simple maps corresponding to the elementary matrices, and 3.4. viii shows any linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be written as elementary matrices. In particular, 3.4. viii (1) - (3) shows any $n \times n$ matrix L can be written as $L = BAC$, where B, C are products of elementary matrices and A is 1 in the top left and $(n-1) \times (n-1)$ bottom right. So an alternative proof is induction on A being able to write as product of elementary matrice using $L = BAC$.

proof. By (3), we have $L = BAC$. The base case is $A = [\lambda]$, which is an elementary matrix. We assume A_{n-1} can be written as a product of elementary matrix. For inductive step, we have

$$A_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{A_{n-1}} & \\ 0 & & & \end{bmatrix},$$

where $A_{n-1} = \prod_{k=1}^{m} E_{n-1}^{k}$, where E_{n-1}^{k} are $(n-1)$-dimensional elementary matrices. But this implies

$$A_n = \prod_{k=1}^{m} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{E_{n-1}^{k}} & \\ 0 & & & \end{bmatrix},$$

which concludes the induction. So now by 3.4.5

$$\deg f_L = (\deg f_B)(\deg f_A)(\deg f_C)$$

Again, if $\det Df_L > 0$, then 0 or 2 of $\det Df_B$, $\det Df_A$, $\det Df_C$ are negative, so 0 or 2 of $\deg f_B$, $\deg f_A$, and $\deg f_C$ are negative, yielding $\deg f_L = 1$, and vice versa. $\square$

3.5 The change of variables formula

- Let U and V be connected open subsets of $\mathbb{R}^n$. If $f: U \to V$ is a diffeomorphism, the determinant of $Df(x)$ at $x \in U$ is nonzero. Since $g = \det(Df(x))$ is continuous, $U \subseteq g^{-1}(x>0) \cup g^{-1}(x<0)$, this shows $Df(x)$ does not flip the sign.
- We say f is **orientation preserving** if this sign is positive and **orientation reversing** if it is negative.

Theorem. The degree of f is $+1$ if f is orientation preserving and -1 if f is orientation reversing.

proof. Given $a_1 \in U$, $a_2 = -f(a_1)$, $g_1 : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + a_1$, $g_2 : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + a_2$

By 3.4.2, $\int_U f^* \omega = \deg f \int_V \omega$, and by 3.4.iv, $\deg g_i = 1$. Thus, the composite diffeomorphism $g_2 \circ f \circ g_1$ has the same degree as f. Note $g_2 \circ f \circ g_1(0) = g_2 \circ f(a_1) = g_2(-a_2) = 0$.

By Theorem 3.4.6 and 3.4.ix, for a bijective linear mapping $A: \mathbb{R}^n \to \mathbb{R}^n$ $\deg A = 1$ if $\det A > 0$ and $\deg A = -1$ otherwise.

Now letting $A = Df_1(0)$, $f_1 = g_2 \circ f \circ g_1$, then

$$D(A^{-1} \circ f_1)(0) = A^{-1} \circ Df_1(0) = id.$$

Now take $f_2 = A^{-1} \circ f_1$, we have $f_2(0) = 0$, $Df_2 = id$.

Let $g(x) = f_2(x) - x$. we have $g(0) = 0$, $Dg(0) = Df_2(0) - 1 = 0$.

For the next step of the proof, we are going to show:

Lemma: There exists a $\delta > 0$ such that $|g(x)| \le \frac{1}{2}|x|$ for $|x| \le \delta$.

proof. Let $g(x) = (g_1(x), \dots, g_n(x))$. Then because $Dg(0) = 0$:

$$\frac{\partial g_i}{\partial x_j}(0) = 0.$$

So there exists a $\delta > 0$ such that $\left| \frac{\partial g_i}{\partial x_j}(x) \right| \le \frac{1}{2n}$ for $|x| < \delta$

So by Mean value theorem,

$$g_i(x) = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}(c)\, x_j, \quad c = t_0 x, \quad 0 < t_0 < 1.$$

Thus, for $|x| < \delta$,

$$|g_i(x)| = \left| \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j}(c)\, x_j \right| \le \sum_{j=1}^{n} \left| \frac{\partial g_i}{\partial x_j}(c)\, x_j \right| \le \frac{n}{2n} |x_j| \le \frac{|x|}{2}.$$

$\square$

Now let ρ be a compactly supported C^∞ function with $0 \le \rho \le 1$ and with $\rho(x) = 0$ for $|x| \ge \delta$, $\rho(x) = 1$ for $|x| \le \frac{\delta}{2}$. Let

$$\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$$

$$x \mapsto x + \rho(x) g(x).$$

So

$$\tilde{f}(x) = \begin{cases} x, & |x| \ge \delta \\ x + g(x) = x + f(x) - x = f(x), & |x| \le \frac{\delta}{2}. \end{cases}$$

when $|x| \le \delta$, $|\tilde{f}(x)| \ge |x| - \rho(x)|g(x)|$

$$\ge |x| - |g(x)| \ge |x| - \tfrac{1}{2}|x| = \tfrac{1}{2}|x|$$

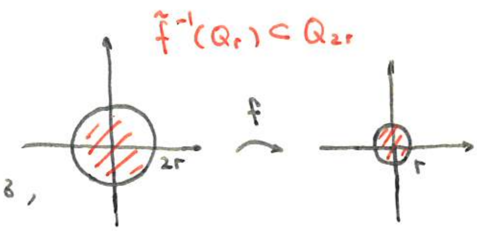
from Lemma above. Therefore, for all $x \in \mathbb{R}^n$,

$$|\tilde{f}(x)| \ge \frac{|x|}{2}.$$

Now let Q_r be the cube $Q_r := \{x \in \mathbb{R}^n \mid |x| \le r\}$, let $Q_r^c := \mathbb{R}^n \setminus Q_r$.

Because $|\tilde{f}(x)| \ge \frac{|x|}{2}$, we have

$$\tilde{f}^{-1}(Q_r) \subset Q_{2r}$$

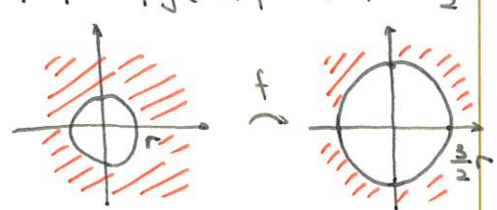


for all r, hence $\tilde{f}$ is proper. For $x \in Q_\delta$,

$$|\tilde{f}(x)| = |x + \rho(x) g(x)| \le |x| + |g(x)| \le |x| + \frac{|x|}{2}$$

by the above Lemma. Therefore,

$$\tilde{f}^{-1}(Q^c_{\frac{3}{2}\delta}) \subset Q^c_\delta$$



$f_2 = A^{-1} \circ f_1$ is a diffeomorphism mapping 0 to 0, it maps a neighborhood U_0 of 0 in U diffeomorphically onto a neighborhood V_0 of 0 in V. Because $\tilde{f}^{-1}(Q_r) \subset Q_{2r}$, we can shrink U_0 to be contained in $Q_{\delta/2}$ and V_0 contained in $Q_{\delta/4}$.

Let ω be an n-form with support in V_0 whose integral over $\mathbb{R}^n$ is equal to 1. Then $f_2^*\omega$ is supported in $U_0 \subset Q_{\delta/2}$, so $f_2^*\omega$ is supported in $Q_{\delta/2}$. Furthermore, $\tilde{f}^{-1}(Q_r) \subset Q_{2r}$ implies that $\tilde{f}^*\omega$ is supported in $Q_{\delta/2}$. Therefore, both $f_2^*\omega$ and $\tilde{f}^*\omega$ are zero outside of $Q_{\delta/2}$ and $\tilde{f} = f_2$ inside $Q_{\delta/2}$. Thus, $f_2 = \tilde{f}$ and

$$\deg f_2 = \int f_2^*\omega = \int \tilde{f}^*\omega = \deg \tilde{f}.$$

Now let $\tilde{\omega}$ be a compactly supported n-form with support in $Q^c_{\frac{3}{2}\delta}$ and $\int \tilde{\omega} = 1$.

$$\tilde{f}^{-1}(Q^c_{\frac{3}{2}\delta}) \subset Q^c_\delta$$

implies $\tilde{f}^*\tilde{\omega}$ supported in Q^c_δ. Since $\tilde{f}(x) = x$ for $|x| \geq \delta$, we have $\tilde{f}^*\tilde{\omega} = \tilde{\omega}$ on Q^c_δ. Thus

$$\deg \tilde{f} = \int \tilde{f}^*\tilde{\omega} = \int \tilde{\omega} = 1.$$

Therefore, $\deg f_2 = 1$. Since $f_2 = A^{-1} \circ f_1$, by 3.4.5, we have

$$\deg f_2 = \deg(A^{-1}) \deg f_1 = 1 \qquad (*)$$

Recall that $A = Df_1(0)$, so

- If f_1 is orientation preserving, then $\det(Df_1(0)) > 0 \Rightarrow \det(Df_1^{-1}(0)) > 0$. Therefore, $\deg(A^{-1}) = 1$ by Theorem 3.4.6.
- If f_1 is orientation reversing, then $\det(Df_1(0)) < 0 \Rightarrow \det(Df_1^{-1}(0)) < 0$. So we have: $\deg(A^{-1}) = -1$.

Therefore, $(*)$ implies $\deg f_1 = 1$ if f_1 is orientation preserving, and $\deg f_1 = -1$ if f_1 is orientation reversing. Finally, we've shown that $\deg f = \deg g_2 \circ f \circ g_1 = \deg f_2$. $\square$

Rapid review on uniform continuity and properties.

Def. For a function $f: X \to Y$ with metric space (X, d_1) and (Y, d_2). f is called uniformly continuous if for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for every $x, y \in X$ with $d_1(x,y) < \delta$, we have $d_2(f(x), f(y)) < \varepsilon$.

Def. A function $f: X \to Y$ with metric space (X, d_1) and (Y, d_2) is called continuous at x if for every real number $\varepsilon > 0$ there exist a real number $\delta > 0$ such that for every $y \in X$ with $d_1(x,y) < \delta$, we have $d_2(f(x), f(y)) < \varepsilon$.

Nonexample.

- Functions unbounded on a bounded domain are not uniformly continuous. For example, $y = \tan x$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$ but not uniformly continuous, as it goes to infinity as $x \to \frac{\pi}{2}$.
- Functions whose derivatives tends to infinity as $x \to \infty$ can not be uniformly continuous. $y = e^x$ is continuous everywhere but not uniformly continuous.
- $f(x) = \sin \frac{1}{x}$ is not uniformly continuous as for any $\delta > 0$, there are $x, y \in (0, 1]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| = 2$.

Heine-Cantor Theorem. If $f: M \to N$ is a continuous function between two metric spaces M and N, and M is compact. Then f is uniformly continuous.

proof. By continuity, for any $\varepsilon > 0$, $\exists \delta_x > 0$ s.t. $d_N(f(x), f(y)) < \frac{\varepsilon}{2}$ when $d_M(x,y) < \delta_x$. Let U_x be the open $\frac{\delta_x}{2}$-neighborhood of x. The collection $\{U_x \mid x \in M\}$ is an open cover of M, thus exist a finite subcover $\{U_{x_k}\}_{k=1}^N$ by compactness. Take $\delta = \min_{1 \le k \le N} \frac{\delta_{x_k}}{2}$. Since N is finite, δ is well-defined and positive. Suppose $d_M(x,y) < \delta$, for any $x, y \in M$. Then $x \in U_{x_i}$ since $\{U_{x_k}\}_{k=1}^N$ is an open cover. So $d_M(x_i, y) \le d_M(x_i, x) + d_M(x, y) \le \delta_{x_i}$. Thus $d_N(f(x), f(y)) \le d_N(f(x), f(x_i)) + d_N(f(x_i), f(y)) < \varepsilon$. ∎

Given $\phi \in C^\infty$, Theorem 3.5.2 is proved using

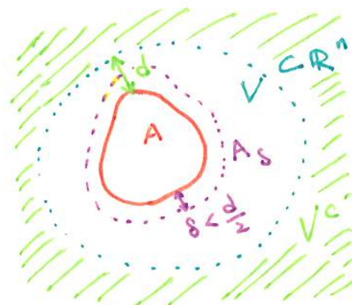
$$\deg f \int_V \phi(y)\,dy = \int_U \phi\circ f(x) \det(Df(x))\,dx$$

$$\Rightarrow \operatorname{sgn}(\det(Df(x))) \int_V \phi(y)\,dy = \int_U \phi\circ f(x) \det(Df(x))\,dx$$

$$\Rightarrow \int_V \phi(y)\,dy = \int_U \phi\circ f(x) \left|\det(Df(x))\right| dx$$

If ϕ is just continuous, we need:

Theorem 3.5.11. Let V be an open subset of $\mathbb{R}^n$. If $\phi: \mathbb{R}^n \to \mathbb{R}$ is a continuous function of compact support with $\operatorname{supp}\phi \subset V$, then for every $\varepsilon > 0$ there exists a C^∞ function of compact support $\psi: \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp}\psi \subset V$, and $\sup|\psi(x) - \phi(x)| < \varepsilon$.



proof. Let A be $\operatorname{supp}\phi$, and let

$$d = \inf_{\substack{x \in V^c \\ y \in A}} \|x - y\|_\infty$$

Since ϕ is continuous and compactly supported, by Heine-Cantor theorem, ϕ is uniformly continuous, i.e., for every $\varepsilon > 0$, $\exists\, \delta > 0$ and $\delta < \frac{d}{2}$ such that $|\phi(x) - \phi(y)| < \varepsilon$ when $|x - y| \le \delta$.

Let Q be the cube $|x| < \delta$ and $\rho: \mathbb{R}^n \to \mathbb{R}$ be a non-negative C^∞ function with $\operatorname{supp}\rho \subset Q$ and

$$\int \rho(y)\,dy = 1.$$

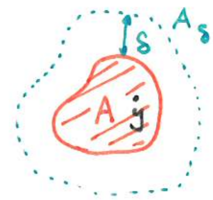
Set

$$\psi(x) = \int \rho(y - x)\,\phi(y)\,dy$$

By Theorem 3.2.10, if all the partial derivatives of $f(x,y)$ with respect to x of order $\le r$ exist and are continuous as functions of x and y, then the function $g(x) = \int f(x,y)\,dy$ is of class C^r. Now we take $f(x,y) = \rho(y - x)\phi(y)$ and $\frac{\partial^r}{\partial x^r} f(x,y)$ exists for all r and are continuous function of x and y.

Therefore, $\psi(x)$ is of class C^r for any r, so $\psi(x) \in C^\infty$.

Let A_δ denote the set of points in $\mathbb{R}^d$ such that:



$$A_\delta := \{ x \in \mathbb{R} \mid \inf_{y \in A} |x-y| \le \delta \}$$

Then for $x \notin A_\delta$ and $y \in A$, $|x-y| > \delta$ and hence $\rho(y-x) = 0$.
Thus, for $x \notin A_\delta$,

$$\int \rho(y-x)\phi(y)\,dy = \int_A \rho(\overbrace{y-x}^{>\delta})\phi(y)\,dy = 0.$$

$\rightarrow$ $\operatorname{supp} \rho \subset Q := \{ x \mid |x| < \delta \}$

So ψ is supported on the compact set A_δ. Since $\delta < \frac{d}{2}$, $\operatorname{supp}\psi$ is contained in V (see the picture in the previous page). We have:

3.4.iv. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the translation $f(x) = x + a$. Show that $\deg f = 1$.

proof. We prove by the fact that for some compactly supported C^∞ function,

$$\int_{\mathbb{R}} \psi(t)\,dt = \int_{\mathbb{R}} \psi(t-a)\,dt,$$

i.e., translation does not change the value of integration over $\mathbb{R}$.

Thus, set $\omega = \phi(x)\,dx$, $\phi(x) = \psi(x_1)\cdots\psi(x_n)$. Then

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \psi(x_1)\cdots\psi(x_n)\,dx_1 \wedge \cdots \wedge dx_n.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^{n-1}} \left(\left[\int_{\mathbb{R}} \psi(x_1)\,dx_1 \right] \psi(x_2) \cdots \psi(x_n) \right) dx_2 \wedge \cdots \wedge dx_n$$

We also have

$$\int f^*\omega = (\phi \circ f)(x) \det(Df(x))\,dx$$

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 + a_1 \\ \vdots \\ x_n + a_n \end{pmatrix},$$ so the Jacobian is $$Df(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

$$\int f^*\omega = \int \phi(x+a)\,dx = \int \psi(x_1+a_1)\cdots\psi(x_n+a_n)\,dx$$

By using Fubini's theorem on each $x_i + a_i$, we have

$$\int \psi(x_i + a_i)\,dx_i = \int \psi(x_i)\,dx_i.$$

Thus, $\int f^*\omega = \int \omega = \deg f \int \omega \Rightarrow \deg f = 1.$ $\square$

Using 3.4.iv and $\int \rho(y)\,dy = 1$, we have

$$\int \rho(y-x)\,dy = \int \rho(y)\,dy = 1.$$

Thus,

$$\phi(x) = \phi(x)\int \rho(y)\,dy = \phi(x)\int \rho(y-x)\,dy = \int \phi(x)\rho(y-x)\,dy$$

And

$$\phi(x) - \psi(x) = \left(\int \phi(x)\rho(y-x)\,dy\right) - \left(\int \phi(y)\rho(y-x)\,dy\right)$$

$$= \int (\phi(x) - \phi(y))\rho(y-x)\,dy.$$

$$|\phi(x) - \psi(x)| \le \int |\phi(x) - \phi(y)|\rho(y-x)\,dy$$

Since $\rho(y-x) = 0$ for $|x-y| \geq \delta$ and $|\phi(x) - \psi(x)| < \varepsilon$ for

$\operatorname{supp}\rho \subset Q := \{x \mid |x| < \delta\}$

$|x-y| \le \delta$ from uniform continuity, so

$$\int |\phi(x) - \phi(y)|\rho(y-x)\,dy < \varepsilon \int \rho(y-x)\,dy.$$

Since $\int \rho(y-x)\,dy = 1$, we conclude

$$|\phi(x) - \psi(x)| < \varepsilon.$$ □

Theorem 3.5.2. Let $\phi : V \to \mathbb{R}$ be a compactly supported continuous function. Then

$$\int_U (\phi \circ f)(x)\,|\det(Df(x))| = \int_V \phi(y)\,dy.$$

proof. Let $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ cut-off function which is one on a neighborhood V_1 of the support of ϕ. γ is non-negative and compactly supported with $\operatorname{supp}\gamma \subset V$. Let

$$c = \int \gamma(y)\,dy. \qquad (\#)$$

By the previous theorem, we have for every $\varepsilon > 0$, a C^∞ function ψ with support V_1 such that

note the theorem can restrict to V_1

$$|\phi - \psi| \le \frac{\varepsilon}{2c}. \qquad (*)$$

So $\left|\int_V (\phi - \psi)(y)\,dy\right| = \left|\int_{\operatorname{supp}\phi \,\cup\, \operatorname{supp}\psi} (\phi - \psi)(y)\,dy\right| \le \int_{\operatorname{supp}\phi \,\cup\, \operatorname{supp}\psi} |(\phi - \psi)(y)|\,dy$

by triangle inequality.

Since V_1 is the neighborhood for the support of ϕ and ψ,

$$\int_{\operatorname{supp}\phi \cup \operatorname{supp}\psi} |(\phi-\psi)(y)\,dy| = \int_{V_1} |(\phi-\psi)(y)\,dy|$$

Note that $\gamma = 1$ on V_1, so

$$\int_{V_1} |(\phi-\psi)(y)\,dy| = \int_{V_1} \gamma(y)\,|(\phi-\psi)(y)\,dy|$$

$$\leq \int_V \gamma(y)\,|(\phi-\psi)(y)\,dy|$$

$$\overset{\text{by }(*)}{\leq} \left(\int_V \gamma(y)\right)\frac{\varepsilon}{2c} \overset{\text{by }(\#)}{=} c\cdot\frac{\varepsilon}{2c} = \frac{\varepsilon}{2}.$$

This shows $\left|\int_V \phi(y)\,dy - \int_V \psi(y)\,dy\right| < \frac{\varepsilon}{2}$. (3.5.15)

Similarly,

$$\left|\int_U (\phi-\psi)\circ f(x)\,|\det Df(x)|\,dx\right|$$

$$\leq \int_U \left|(\phi-\psi)\circ f(x)\,|\det Df(x)|\,dx\right|$$

$$= \int_{\{x \mid f(x)\in \operatorname{supp}\phi \cup \operatorname{supp}\psi\}} \left|(\phi-\psi)\circ f(x)\,|\det Df(x)|\,dx\right|$$

$$= \int_{\{x \mid f(x)\in V_1\}} \left|(\phi-\psi)\circ f(x)\,|\det Df(x)|\,dx\right|$$

$$= \int_{\{x \mid f(x)\in V_1\}} \gamma\circ f(x)\left|(\phi-\psi)\circ f(x)\,|\det Df(x)|\,dx\right|$$

$$\leq \int_U \gamma\circ f(x)\left|(\phi-\psi)\circ f(x)\,|\det Df(x)|\,dx\right|$$

$$< \frac{\varepsilon}{2c}\int_U \gamma\circ f(x)\,|\det Df(x)|\,dx$$

$$= \frac{\varepsilon}{2c}\int_V \gamma(y)\,dy$$ by (3.5.3): $\int_U (\phi\circ f)(x)\,|\det Df(x)| = \int_V \phi(y)\,dy$

$$= \frac{\varepsilon}{2}$$

This shows

$$\left| \int_U \phi \circ f(x)\, |\det Df(x)|\, dx - \int_U \psi \circ f(x)\, |\det Df(x)|\, dx \right| < \frac{\varepsilon}{2} \qquad (3.5.16)$$

Also, combining

$$\int_V \psi(y)\, dy = \int_U \psi \circ f(x)\, |\det Df(x)|\, dx$$

with (3.5.15) and (3.5.16) gives

$$\left| \int_V \phi(y)\, dy - \int_U \phi \circ f(x)\, |\det Df(x)|\, dx \right|$$

$$= \left| \int_V \phi(y)\, dy - \int_V \psi(y)\, dy + \int_U \psi \circ f(x)\, |\det Df(x)|\, dx - \int_U \phi \circ f(x)\, |\det Df(x)|\, dx \right|$$

$$\leq \left| \int_V \phi(y)\, dy - \int_V \psi(y)\, dy \right| + \left| \int_U \psi \circ f(x)\, |\det Df(x)|\, dx - \int_U \phi \circ f(x)\, |\det Df(x)|\, dx \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon .$$

This concludes

$$\int_V \phi(y)\, dy = \int_U \phi \circ f(x)\, |\det Df(x)|\, dx. \qquad \square$$

3.6 Techniques for computing the degree of a mapping

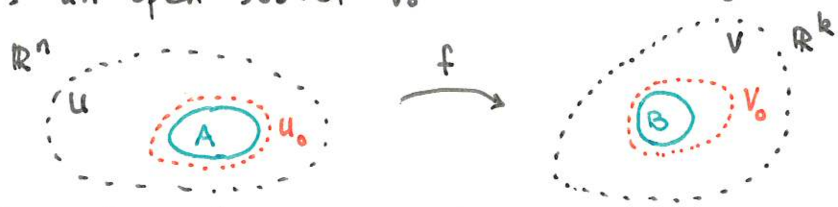
- This chapter discusses how to compute the degree of f, which is always an integer. The degree of f is a topological invariant: if we deform f smoothly, the degree doesn't change.

Def. A point $x \in U$ is a critical point of f if the derivative

$$Df(x): \mathbb{R}^n \to \mathbb{R}^n$$

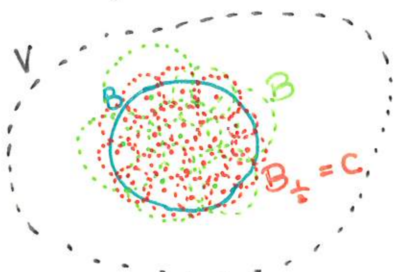
fails to be bijective, i.e., if $\det(Df(x)) = 0$.

Theorem 3.4.7. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$. Let $f: U \to V$ be a proper continuous mapping. If B is a compact subset of V and $A = f^{-1}(B)$, then for every open subset U_0 with $A \subset U_0 \subset U$, there exists an open subset V_0 with $B \subset V_0 \subset V$ and $f^{-1}(V_0) \subset U_0$.



proof. First we prove there exists a compact subset C of V with $B \subset \text{int } C$.

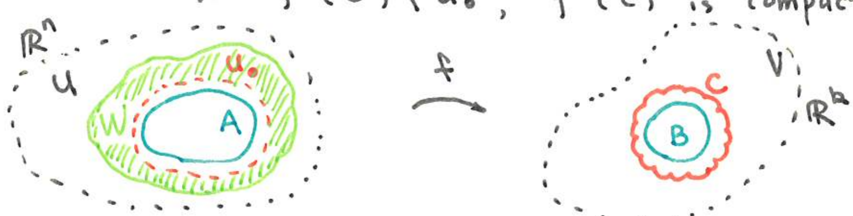
To see this, first note for each $x \in B$ has a corresponding ball of radius $r_x > 0$ such that $B_{r_x}(x) \subset V$ since V is open.



Inside $\mathcal{B} = \bigcup_x B_{r_x}(x)$ is $\bigcup_x B_{\frac{r_x}{2}}(x) = \mathcal{B}_{\frac{1}{2}}$ and it has a finite subcover since B is compact.

Then $C = \mathcal{B}_{\frac{1}{2}}$ since it contains B and in V.

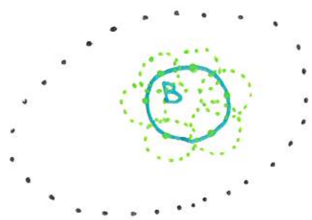
Then we take $W = f^{-1}(C) \setminus U_0$, $f^{-1}(C)$ is compact because f is proper.



So W is compact and $f(W)$ is compact by continuity of f. Set $V_0 = \text{int } C \setminus f(W)$ strictly contains B and contained (strictly) by V. □

Alternative proof for the existence of C:

proof. Start with an open set V and a compact set $B \subset V$. Then each point $x \in \partial B$ has a neighborhood of radius r_x that is contained in V. Since B is compact, ∂B can be covered by finitely many subcovers,



denote this subcover by $\mathcal{B}$. Then take $r = \min_{x \in \mathcal{B}} r_x \neq 0$ since $\mathcal{B}$ is a finite subcover. Then take $V_0 = \bigcup_{x \in \partial B} B_r(x) \cup B$. □

3.4. iii. Show that if $f: U \to V$ is a proper continuous mapping and X is a closed subset of U. Then $f(X)$ is closed.

proof. Pick $U_0 = U - X$, then if $p \in V \setminus f(X)$, then $f^{-1}(p) \subset U_0$, otherwise if $y \in f^{-1}(p) \subset X$, then $f(y) \subset X$. But $p = f(y)$, contradiction.

By theorem 3.4.7, there exists a neighborhood V_0 of p, such that $f^{-1}(V_0)$ is contained in U_0.

Assume $\exists x \in V_0 \cap f(X)$, then $f^{-1}(x) \in U_0$ since $f^{-1}(V_0) \subset U_0$. But $x \in f(X)$ so for some $z \in f^{-1}(x)$, $z \in X$. But z is also in U_0 because $f^{-1}(x) \in U_0$, contradiction. So V_0 and $f(X)$ are disjoint. This shows every point $p \in V \setminus f(x)$ has an open neighborhood that is disjoint from $f(x)$, concluding $f(x)$ is closed. □

- We denote the set of critical points of f by C_f. C_f is a closed subset of U, since it is the preimage of a closed set $\{0\}$ for the continuous function $\det Df$. Therefore, $f(C_f)$ is a closed subset of V.
- We call this image the set of critical values of f, and the complement of this image the set of regular values of f.

- $V \setminus f(U)$ is contained in $f(U) \setminus f(C_f)$. So if $q \in V$, $q \notin f(U)$, then q is a regular value of f by default.
- Example. C_f can be quite large. If $c \in V$, $f : U \to V$ is the constant map which maps all of U onto c, then $C_f = U$. The regular values of f is $V \setminus \{c\}$, which is an open dense subset of V. This is true in general for any proper C^∞ map, called Sard's theorem.

Theorem 3.6.3. The set $f^{-1}(q)$ is a finite set. If $f^{-1}(q) = \{p_1, \dots, p_m\}$, there exist connected open neighborhoods U_i of p_i in U and an open neighborhood W of q in V such that (for q being a regular value)

(1) for $i \neq j$ the sets U_i and U_j are disjoint.

proof. Assume $\exists p \in f^{-1}(q)$, since q is a regular value, $p \notin C_f$. So

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^n$$

is bijective. By inverse function theorem, f maps a neighborhood U_p of p diffeomorphically onto a neighborhood of q. The open sets

$$\{U_p \mid p \in f^{-1}(q)\}$$

are a covering of $f^{-1}(q)$. Since f is proper, $f^{-1}(q)$ is compact. So

$$\{U_{p_1}, \dots, U_{p_m}\}$$

is a finite subcovering of $f^{-1}(q)$. We can pick U_p small enough so that each U_p only contains one p_i. Also $f(U_p) = q$, so

$$f^{-1}(q) = \{p_1, \dots, p_m\}.$$

This shows $f^{-1}(q)$ is a finite set and U_i and U_j are disjoint for $i \neq j$. $\square$

(2) $f^{-1}(W) = U_1 \cup \cdots \cup U_m$.

proof. By Theorem 3.4.7, we have a connected open neighborhood W of q in V for which

$$f^{-1}(W) \subset U_{p_1} \cup \cdots \cup U_{p_m}$$

Let $U_i := f^{-1}(W) \cap U_{p_i}$ conclude the proof. □

We also have shown (at the beginning):

(3) f maps U_i diffeomorphically onto W.

Computing the Degree of f

Theorem 3.6.4. For each $p_i \in f^{-1}(q)$, let $\sigma_{p_i} = +1$ if $f: U_i \to W$ is orientation-preserving and -1 if $f: U_i \to W$ is orientation reversing. Then

$$\deg(f) = \sum_{i=1}^{m} \sigma_{p_i}.$$

proof. Let ω be a compactly supported n-form on W whose integral is one. Then

$$\deg(f)\int_V \omega = \int_U f^*\omega \quad \text{by definition.}$$

By previous Theorem, $f^{-1}(W) = U_1 \cup \cdots \cup U_m$, so

$$\deg(f) = \sum_{i=1}^{m} \int_{U_i} f^*\omega.$$

$f|_{U_i}: U_i \to W$ is a diffeomorphism by the previous theorem. So

$$\int_{U_i} f_i^*\omega = \deg(f_i)\int_W \omega \quad , \quad f_i := f|_{U_i}.$$

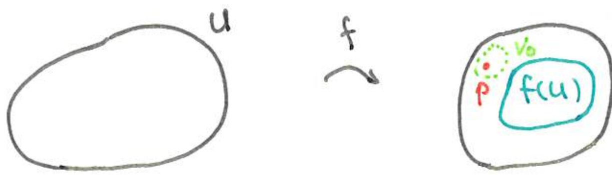
Then by Theorem 3.5.1, the degree of f_i is 1 if f_i is orientation preserving, and -1 if f_i is orientation reversing, given f_i a diffeomorphism. So $\deg(f) = \sum_{i=1}^{m} \sigma_{p_i}$.

- A point $q \in V$ can qualify as a regular value of f by not being in the image of f.

Recall we have proved for a proper continuous mapping $f: U \to V$:

Theorem 3.4.7. If B is a compact subset of V and $A = f^{-1}(B)$ then for every open subset U_0 with $A \subset U_0 \subset U$, there exists an open subset V_0 with $B \subset V_0 \subset V$ and $f^{-1}(V_0) \subset U_0$.

- Now given f proper, pick $p \in V \setminus f(U)$. Take $B = \{p\}$, then $A = \phi$. Pick $U_0 = \phi$, by Theorem 3.4.7, there exists an open subset with $B = \{p\} \subset V_0 \subset V$ and $f^{-1}(V_0) \subset U_0$.



If $f^{-1}(V_0) \subset U_0$, then $f^{-1}(V_0) = \phi$. So $V_0 \subset V \setminus f(U)$. This shows every point p has an open neighborhood. thus $V \setminus f(U)$ is open since p is an arbitrary point in $V \setminus f(U)$.

Theorem 3.6.6. If $f: U \to V$ is not surjective, then $\deg(f) = 0$.

proof. Given $V \setminus f(U)$ being open, if it is non-empty, by bump function construction (A. 4), there exists a compactly supported n-form ω with support in $V \setminus f(U)$ and integral equal to 1. Since $\omega = 0$ on the image of f, we have $f^* \omega = 0$. So

$$0 = \int_U f^* \omega = \deg(f) \int_V \omega = \deg f.$$

Theorem 3.6.7. If $\deg f \neq 0$, then f maps U surjectively onto V.

- We will now show the degree of f is a topological invariant of f.
- Let U be an open subset of $\mathbb{R}^m$, V an open subset of $\mathbb{R}^n$. An open subinterval of $\mathbb{R}$ denoted by $A \ni \{0,1\}$, $f_1, f_2 : U \to V$ are C^∞. Then a C^∞ map $F: U \times A \longrightarrow$ is a homotopy between f_1 and f_2 if $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$.

- Now suppose f_1, f_2 are proper.

Def. A homotopy F between f_0 and f_1 is a proper homotopy if the map followed is proper:

$$F^{\#} : U \times A \longrightarrow V \times A$$

$$(x, t) \mapsto (F(x,t), t).$$

- If F is a proper homotopy between f_0 and f_1, then for every t between 0 and 1, the map followed is proper:

$$f_t : U \longrightarrow V$$

$$f_t(x) : F(x,t)$$

Now let U, V be open subsets of $\mathbb{R}^n$:

Theorem 3.6.10. If f_0 and f_1 are properly homotopic, then $\deg f_0 = \deg f_1$

proof. We construct a form

$$\omega = \phi(y)\, dy_1 \wedge \cdots \wedge dy_n$$

compactly supported n-form on V, $\int_V \omega = 1$.

From (3.4.2) $\int_U f^* \omega = \deg f \int_V \omega$ and

$$f^* \omega = (\phi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n,$$

we have

$$\deg f_t \int_V \omega = \deg f_t = \int_U f_t^* \omega \qquad (*)$$

$$= (\phi \circ f_t)(x) \det(Df_t(x))\, dx_1 \wedge \cdots \wedge dx_n$$

But $\deg f_t \in \mathbb{Z}$ and the integrand is continuous in $t \in [0,1]$, supported on a compact subset of $U \times [0,1]$. Hence $(*)$ is continuous as a function of t, therefore $\deg(f_t)$ is integer valued so the function is a constant. □

Now we prove two lemmas for the Brouwer fixed point theorem.

3.6.i. Let W be a subset of $\mathbb{R}^n$ and $a(x)$, $b(x)$, $c(x)$ be real-valued functions on W of class C^r. Suppose for every $x \in W$, the quadratic polynomial

$$a(x)s^2 + b(x)s + c(x)$$

has two distinct real roots, $s_+(x)$ and $s_-(x)$ with $s_+(x) > s_-(x)$.

Prove s_+ and s_- are functions of class C^r.

proof. C^r is closed under $+, -, \times, \circ$, follows from differentiation rules, and $\frac{f}{g}$ is C^r if g is never zero.

Since there are 2 real roots, $a(x)$ is never zero, and

$$b^2(x) - 4a(x)c(x) > 0$$

So $s_+, s_- = \dfrac{-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x)}}{2a(x)}$. Given there are two distinct roots, $b^2(x) - 4a(x)c(x) > 0$, thus

$$\left(\left(b^2(x) - 4a(x)c(x)\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(b^2(x) - 4a(x)c(x)\right)^{-\frac{1}{2}}\left(b^2(x) - 4a(x)c(x)\right)'$$

remains C^r differentiable.

If W is path connected, then by intermediate value theorem, $a(x) \neq 0$ implies $a(x)$ does not switch signs. So the choice of s_- and s_+ is continuous. If W is not path connected, this is the same for each connected component, so it does not affect the differentiability either.

Therefore, s_+ and s_- are functions of class C^r. $\square$

Applications : The Brouwer fixed point theorem. Let B^n be the closed unit ball in $\mathbb{R}^n$:

$$B^n := \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}.$$

If $f : B^n \to B^n$ is a continuous mapping, the f has a fix point:

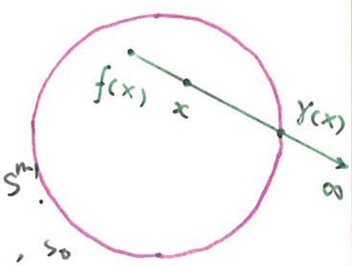
$$\exists\, x_0 \in B^n : f(x_0) = x_0 .$$

※ The idea of the proof is to assume there is no fixed point and lead to a contradiction.

proof. Suppose $\forall x \in B^n$ we have $f(x) \neq x$. Consider the ray through $f(x)$ in the direction of x:

$$f(x) + s(x - f(x)), \quad s \in [0, \infty).$$

The ray intersects the boundary $S^{n-1} := \partial B^n$ in a unique point $\gamma(x)$, and we need a lemma:



3.6.ii. Show that $\gamma(x)$ is a continuous surjection $B^n \to S^{n-1}$.

proof. If $x \in S^{n-1}$, $\gamma(x) = x$, i.e., substitu $s = 1$, so

$$\gamma(x) = f(x) + s(x - f(x)) = f(x) + x - f(x) = x .$$

This shows γ is surjective. Now we show

$$\| f(x) + s(x - f(x))\|^2 - 1 \qquad (*)$$

has two real roots. This is because $\|\gamma(x)\| = 1$, so

$$\gamma(x) = f(x) + s_0 (x - f(x)),$$

where s_0 is the non-negative root of the quadratic polynomial $(*)$. Given $f(x) \neq x$ (otherwise we have a fixed point), the picture above shows that $\gamma(x)$ intersect S^1 at two distinct points (extending the ray to a line). Therefore, using 3.6.i. we know $\gamma(x)$ is continuous if $f(x)$ is continuous. $\square$

Because $\gamma(x) = x$ if $x \in S^{n-1}$, so we can extend γ to a continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ by letting γ be the identity for $\|x\| > 1$. This extended map satisfies

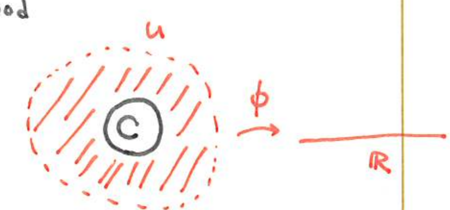
$$\|\gamma(x)\| \geq 1$$

for all $x \in \mathbb{R}^n$ because $\gamma(x) \in S^{n-1}$, so $\|\gamma(x)\| = 1$ before the extension, and $\|\gamma(x)\| = \|x\| > 1$ for $x \in \mathbb{R}^n \setminus \overline{B}_1(0)$. Also, because $\gamma(x) = x$ $\forall x \in S^{n-1}$, this extension is also continuous.

To lead to a contradiction when there isn't a fixed point, we need to show that γ can be approximated by a C^∞ map. To achieve that, we prove the following corollary of Theorem 3.5.11.

Corollary 3.6.15. Let U be an open subset of $\mathbb{R}^n$, C a compact subset of U and $\phi : U \to \mathbb{R}$ a continuous function which is C^∞ on the complement of C. Then for every $\varepsilon > 0$, there exists a C^∞ function $\psi : U \to \mathbb{R}$, such that $\phi - \psi$ has compact support and $|\phi - \psi| < \varepsilon$.

proof. Let ρ be a bump function in $C_0^\infty(U)$, where C_0 refers to vanish at infinity, and is equal to 1 on a neighborhood of C. By Theorem 3.5.11, $\exists\, \psi_0 \in C_0^\infty(U)$ that



$$|\rho\phi - \psi_0| < \varepsilon.$$

Let $\psi := (1-\rho)\phi + \psi_0$, so

$$\phi - \psi = \phi - \phi + \rho\phi - \psi_0 = \underbrace{\rho\phi}_{\text{compact}} - \underbrace{\psi_0}_{\text{compact}}. \qquad \square$$

Let $g = (g_1, \dots, g_n)$, where $g_i : \mathbb{R}^n \to \mathbb{R}$ $\in C^\infty$ according to Corollary 3.5.15 with each coordinate of map γ with g_i, then we have

$$\|g - \gamma\|_\infty < \varepsilon < 1, \text{ by choosing } \varepsilon < 1.$$

By Corollary 3.6.15, $g_i - \gamma_i$ has compact support. Since $g, \gamma : \mathbb{R}^n \to \mathbb{R}^n$, this implies $g_i - \gamma_i = 0$ outside of some compact set C_i. Therefore,

$$\| g - \gamma \|_\infty = 0 \quad \text{outside of some compact set } \bigcup_{i=1}^{n} C_i .$$

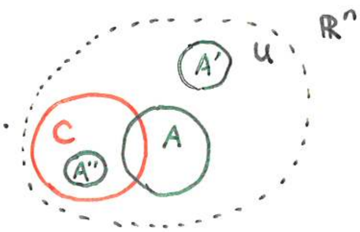
Therefore, $g = \gamma$ on $\mathbb{R}^n \setminus \bigcup_{i=1}^{n} C_i$. Since $\gamma(x) = x$ on $\mathbb{R}^n \setminus B^n_{r_i}(0)$, so

$$g(x) = x \quad \text{on some compact set } \mathbb{R}^n \setminus \left(\bigcup_{i=1}^{n} C_i \cup B^n_{r_i}(0) \right).$$

Now we need the following lemma:

3.6.ix. Let U be an open connected subset of $\mathbb{R}^n$ and $f : U \to U$ any C^∞ map. Prove that if f is equal to the identity on the complement of a compact set C, then f is proper and $\deg f = 1$.

proof. For any subset $A \subset U$, since f is the identity on $U \setminus C$, then $f^{-1}(A) \subset A \cup C$.



A, A', A'' are different choices of A

Since f is continuous, to show f is proper, suffice to show pre-image of bounded set is bounded. (The pre-image of a closed set is closed comes from f being continuous.) But if A is bounded, then $f^{-1}(A) \subset A \cup C$ is also bounded since C is compact.

For the next part, we consider a point $q \in U \setminus f(C)$, so

$$f^{-1}(q) = \{q\}$$

and identity is orientation preserving. So by the degree formula 3.6.5,

$$\deg f = 1 .$$

Such a q must exist since $f(C)$ is compact. If $f(C) = U$, then U is clopen, i.e., $U = \mathbb{R}^n$, contradiction. $\square$

Now return to the proof of Brouwer fixed point theorem, by 3.6.ix, g is proper and $\deg g = 1$. So g must be surjective by Theorem 3.6.8.

However, $\|g - \gamma\| < \varepsilon \overset{\|\gamma\| \geq 1}{\Rightarrow} \|\gamma\| - \|g-\gamma\| > 1 - \varepsilon \overset{\Delta\text{-ineq}}{\Rightarrow} \|\gamma\| - \|\gamma\| + \|g\| > 1 - \varepsilon \Rightarrow 0 \notin \operatorname{im}(g)$. $\square$

Application 3.6.16 (The fundamental theorem of algebra). Let

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree n with complex coefficients. If we identify the complex plane

$$\mathbb{C} := \{ z = x + iy \mid x, y \in \mathbb{R} \}$$

with $\mathbb{R}^2$ via the map $\mathbb{R}^2 \to \mathbb{C}$ given by $(x, y) \mapsto z = x + iy$, we can think of p as identifying a mapping

$$p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$z \mapsto p(z).$$

Theorem. The mapping $p : \mathbb{R}^2 \to \mathbb{R}^2$ is proper and $\deg(p) = n$.

proof. For $t \in \mathbb{R}$, let

$$p_t(z) := (1-t) z^n + t p(z) = z^n + t \sum_{i=0}^{n-1} a_i z^i.$$

Now we show the mapping

$$g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$$
$$(z, t) \mapsto p_t(z)$$

is a proper homotopy. Let

$$C = \sup_{0 \le i \le n-1} |a_i|,$$

then for $|z| \geq 1$, we have

$$\left| \sum_{i=0}^{n-1} a_i z^i \right| \le \sum_{i=0}^{n-1} |a_i| |z|^i \le C n |z|^{n-1} \qquad (*)$$

If $|t| \le a$ and $|z| \geq 2aCn$, we have

$$\begin{aligned}
|p_t(z)| &= \left| z^n + t \sum_{t=0}^{n-1} a_i z^i \right| \\
&\geq |z|^n - \left| t \sum_{t=0}^{n-1} a_i z^i \right| && \text{from } |A+B+(-B)| \le |A+B| + |B| \\
&\geq |z|^n - a \left| \sum_{t=0}^{n-1} a_i z^i \right| && \text{from } |t| \le a \\
&\geq |z|^n - a \sum_{i=0}^{n-1} |a_i| |z|^i && \text{from above } (*) \\
&\geq |z|^n - a \cdot Cn |z|^{n-1} \geq 2aC_n |z|^{n-1} - a \cdot Cn |z|^{n-1}
\end{aligned}$$

So we can conclude that

$$|P_t(z)| \geq |z|^n - a\,C\,n\,|z|^{n-1}$$
$$\geq a\,C\,n\,|z|^{n-1}$$

If $A \subset \mathbb{C}$ is compact, then for some $R>0$, A is contained in $|w| \leq R$. So

$$\{z \in \mathbb{C} \mid (t, P_t(z)) \in [-a,a] \times A\} \subset \{z \in \mathbb{C} \mid a\,C\,|z|^{n-1} \leq R\}.$$

This shows $g(z,t) = P_t(z)$ is a proper homotopy. Thus, for each $t \in \mathbb{R}$, the map $P_t : \mathbb{C} \to \mathbb{C}$ is proper and

$$\deg(P_t) = \deg(P_1) = \deg(P_0) = \deg(P).$$

However, $P_0 : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ is an elementary computation with degree n.
(see 3.6.v, 3.6.vi)

3.6.v. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via

$$\mathbb{C} \longrightarrow \mathbb{R}^2$$
$$x + iy \mapsto (x,y)$$

we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself a mapping:

$$z \mapsto cz$$

for a fixed $c \in \mathbb{C}$ (as an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself). Show the determinant of this mapping is $|c|^2$.

proof. Take $m : \mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto cz$, then $m(x+iy) = c(x+iy)$; set $c = a+bi$, then

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax-by \\ ay+bx \end{bmatrix} \Rightarrow m = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \det m = a^2 + b^2 = |c|^2. \qquad \square$$

3.6. vi. (1) Let $f: \mathbb{C} \to \mathbb{C}$ be the mapping $f(z) = z^n$. Show $Df(z)$ is the linear map

$$Df(z) = n z^{n-1}$$

given by multiplication by $n z^{n-1}$.

proof. $$Df(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{(z+h)^n - z^n}{h}$$

$$= \lim_{h \to 0} \frac{n h z^{n-1} + O(h^2) + z^n - z^n}{h} = n z^{n-1}.$$

(2) Conclude from 3.6. v that

$$\det(Df(z)) = n^2 |z|^{2n-2}.$$

proof. Let $n z^{n-1} = a + bi$, $Df(z) = (a+bi)(x+yi)$

$$Df(z) = m z = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}$$

$$\Rightarrow m = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \det m = \det(Df(z)) = a^2 + b^2 = |n z^{n-1}|^2.$$

(3) Show that at every point $z \in \mathbb{C} \setminus \{0\}$, f is orientation preserving.

proof. Given $\det(Df(z)) = n^2 |z|^{2n-2}$, we have $\det(Df(z)) > 0$, here we assumed $n > 0$. Therefore, f is orientation preserving by definition.

(4) Show that every point $w \in \mathbb{C} \setminus \{0\}$ is a regular value of f and

$$f^{-1}(w) = \{z_1, \dots, z_n\}$$

with $\sigma_{z_i} = +1$.

proof. Because every point $w \in \mathbb{C} \setminus \{0\}$ has $\det(Df(z)) > 0$, so their images are regular values (as well as points not in the image). The only preimage of 0 is 0, so every point in $\mathbb{C} \setminus \{0\}$ is a regular value. $f^{-1}(w) = \{z_1, \dots, z_n\}$ follows from roots of unity, and $\sigma_{z_i} = +1$ follows from f is orientation preserving for points in $\mathbb{C} \setminus \{0\}$.

(5) Conclude the degree of f is n.

proof. Using Theorem 3.6.4, we have $\deg(z^n) = \sum_{i=1}^{m} \delta_{p_i} = m$. $\square$

3.7 Sard's Theorem

Theorem. Let U be an open subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ a C^∞ map. Then $\mathbb{R}^n \setminus f(C_f)$ is dense in $\mathbb{R}^n$.

Remark. f needs to be proper in order for $\mathbb{R}^n \setminus f(C_f)$ to be open.

A stronger version:

Theorem. Let $f: U \to V$ be a smooth map of manifolds. Then $f(C_f)$ has measure zero in V.

Remark. It is possible for $\mathbb{R}^n \setminus f(C_f)$ to be dense, yet $f(C_f)$ is not measure zero. For example, $\mathbb{Q} \subset \mathbb{R}$ is dense, but $\mathbb{R} \setminus \mathbb{Q}$ does not have measure zero.

Motivation

Recall we defined for $f: U \to V$, $U, V \subseteq \mathbb{R}^n$,

critical points $C_f := \{ p \mid \det(Df(p)) = 0 \}$ i.e., $Df(p)$ isn't full rank.

critical values $f(C_f)$

regular values $V \setminus f(C_f)$

Regular values are much nicer to generalize, and Sard's theorem says we have a lot of regular values to take. For example, regular values give diffeomorphisms (inverse function theorem).

Definition. Let A be open in $\mathbb{R}^k$. A function $f: A \to \mathbb{R}^k$ is a diffeomorphism of A onto its image $B = f(A)$ if it is one-to-one, smooth, of full rank k.

Theorem (Inverse Function Theorem) Let A be open in $\mathbb{R}^k$, $f: A \to \mathbb{R}^k$ be a smooth function. If for $x_0 \in A$, $Df(x_0)$ is of rank k, then there exists an open neighborhood U of x_0 that f is a one-to-one mapping of U onto its image $f(U)$ and the inverse f^{-1} is a smooth function on $f(U)$

- In 3.6 the following corollary of Sard's theorem was used:

Corollary. f has a regular value.

- Q: Why this is a difficult statement?
- A: There is an imbalance between critical and regular values.

If ANY of v's preimage is critical, then v is a critical value.

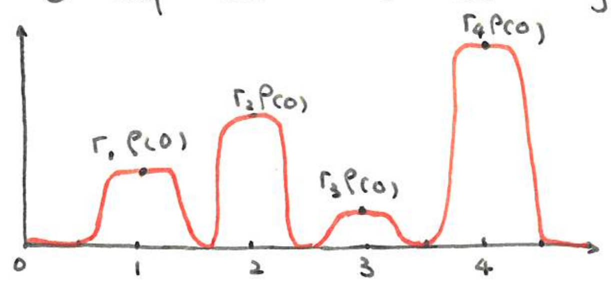
If NONE of v's preimage is critical, then v is a regular value.

i.e., in a sense, it is easy to make a $v \in V$ critical.

Eg. 3.7.iii. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function which is supported in the interval $(-\frac{1}{2}, \frac{1}{2})$ and has a maximum at the origin. Let $(r_i)_{i \geq 1}$ be an enumeration of the rational numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be the map

$$f(x) = \sum_{i=1}^{\infty} r_i \rho(x-i), \quad \text{i.e., a piece-wise function.}$$

Show that f is a C^∞ map and show the image of C_f is dense in $\mathbb{R}$.



proof. As ρ has a max at 0, $0 \in C_f$. Since $\operatorname{supp} \rho = (-\frac{1}{2}, \frac{1}{2})$, $\forall k \in \mathbb{N}$, $x \in (k - \frac{1}{2}, k + \frac{1}{2})$, we have

$$f(x) = r_k \rho(x-k).$$

Otherwise, $f(x) = 0$. So f is C^∞ and the critical points are $x = k$, for $k \in \mathbb{N}$. Hence,

$$f(\mathbb{N}) = \rho(0)\mathbb{Q} \subseteq f(C_f).$$

Since $\mathbb{Q}$ are dense in $\mathbb{R}$, $f(C_f)$ is dense as well, given $\rho(0) \neq 0$. $\square$

Remark. Sard's Theorem tells the complement of the critical values are dense, but we can construct an f such that the critical values are any canonical setting, such as the rationals.

Proof Outline of Sard's Theorem

Equivalent to Sard's theorem

1. Enough To Show: $\mathbb{R}^n \setminus \boxed{f(C_f \cap A)}$ is dense $\forall$ compact A. $\subseteq f(C_f)$
   - Specifically, n-dim closed cubes A.
   - This also corresponds to 3.7.2, 3.7.3, ex. 3.7. iv.
   - Baire category theorem (3.7.2)

will prove $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense

2. Motivating example for induction set-up.
   - Cases for rank $(Df(p))$
   - Motivation for super-critical points of f:

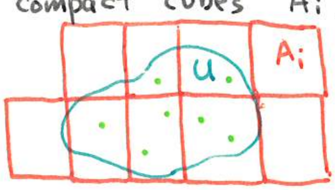
     $C_f^{\#} := \{p \in U \mid Df(p) = 0\}$. (i.e., rank $(Df(p)) = 0$)

3. Proof that $\mathbb{R}^n \setminus \boxed{f(C_f^{\#} \cap A)}$ is dense for closed n-dim cube A. $\subseteq f(C_f \cap A)$
4. Induction step on the dimension n
   - Reducing dimension n and rank of $Df(p)$.
   - 3.7.4, 3.7. iv
   - A lot of use of BCT.

Alternative proof: Ryan Blair MATH 600 lecture notes

Step 1. It suffices to prove $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense $\forall$ compact A.

Basic reason: we can cover U with a countable collection of closed and compact cubes A_i. So $f(C_f) = \bigcup_i f(C_f \cap A_i)$



parceling out critical points

So $\mathbb{R}^n \setminus f(C_f) = \bigcap_i \mathbb{R}^n \setminus f(C_f \cap A_i)$. (*)

The claim will follow from the next two remarks.

Rmk 3.7.2 (BCT). If $(U_m)_{m \geq 1}$ are open dense subsets of $\mathbb{R}^n$, the intersection $\bigcap_m U_m$ is dense in $\mathbb{R}^n$.

Rmk 3.7.3. If A is a compact set, then $\mathbb{R}^n \setminus f(C_f \cap A)$ is open.

Remark 3.7.2, 3.7.3 and (*) together will prove Step 1, since if we proved $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense, then it is open (Rmk 3.7.3). Thus, by BCT, their intersection $\bigcap_i \mathbb{R}^n \setminus f(C_f \cap A_i)$ is also dense. So we will prove Sard's theorem.

<u>proof of Rmk 3.7.2</u>. We must show $\forall p \in \mathbb{R}^n$ and neighborhood V of p, $V \cap \bigcap_m V_m$ is non-empty. We will iteratively construct a sequence of balls $B_k := B(x_k, r_k)$ such that

$$B_k \subset V \cap \left(\bigcap_{m \le k} U_m\right).$$

We also require the balls getting smaller to allow an argument about Cauchy sequence: $r_k < \frac{1}{k}$. We will also show those balls contain each other (and their radii are getting smaller).

Initially, as U_1 and V are open, so $V \cap U_1$ is open. Also, U_i is dense, so we can find point x_1 and radius $r_1 < 1$ such that

$$\bar{B}(x_1, r_1) \subseteq V \cap U_1.$$

We can continuous this construction. If we have B_{k-1}, then

$$B_{k-1} \cap U_k$$

will again be non-empty and open. (The continuous choice of the balls depends on the axiom of choice.) Then we have x_k, $r_k < \frac{1}{k}$ that

$$\bar{B}(x_k, r_k) \subseteq B_{k-1} \cap U_k.$$

Then (x_n) is a Cauchy sequence as the balls contains each other and have $r_k \to 0$. So it converges to some $x \in \bigcap_{k \ge 1} B_k$. Thus, $x \in V$ and $x \in U_m$, $\forall m$. This proves the the intersection of $V \cap (\bigcap_{m \le k} U_m)$ is not empty.

$\square$

<u>proof of 3.7.3</u>. Recall that C_f is closed by definition, since it is the preimage of a point. As f is smoth, the intersection $f(C_f \cap A)$ is compact since A is compact. So

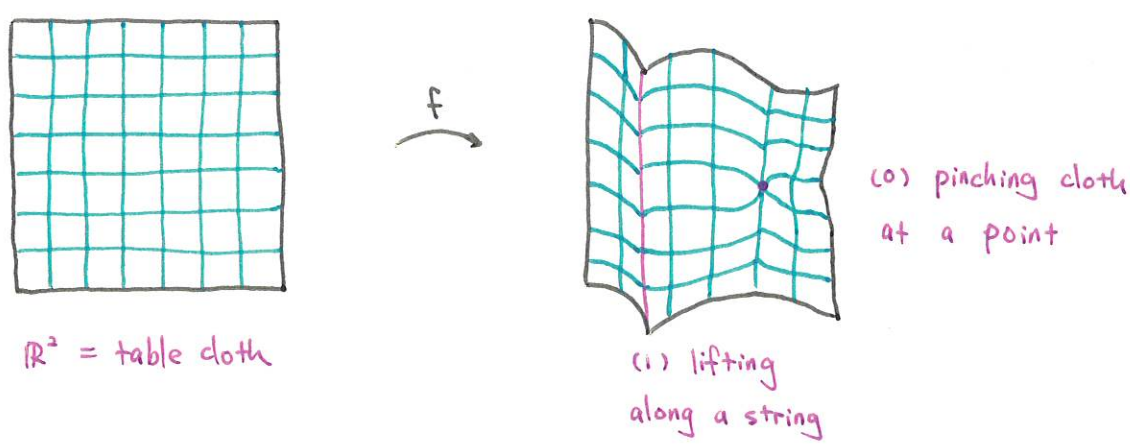
$$\mathbb{R}^n \setminus f(C_f \cap A)$$

is open. □

Remark. If U_m are not open, then BCT is false. For example, $Q \cap (Q + \sqrt{2})$.

Remark. In 3.6, f is proper and C^∞. In this section, f is just C^∞. So $\mathbb{R}^n \setminus f(C_f)$ may not be open. For example, by 3.4.iii, if $f: U \to V$ is a proper continuous mapping and X is a closed subset of U, then $f(X)$ is closed. If f is not proper, then $f(C_f)$ may not be closed, for example, the exercise with $\mathbb{Q} \subseteq f(C_f)$. If we restrict ρ attains a unique max value, such as the bomp function and set $\rho(0) = 1$, then $f(C_f) = \mathbb{Q}$. In this case, $\mathbb{R}^n \setminus \mathbb{Q}$ is the rationals and are not open.

<u>Step 2.</u> Critical points are $p \in U$ such that $Df(p)$ is not full rank. Our proof will handle critical points differently depending on the rank. To provide some intuition, consider the following example for $f: \mathbb{R}^2 \to \mathbb{R}^2$

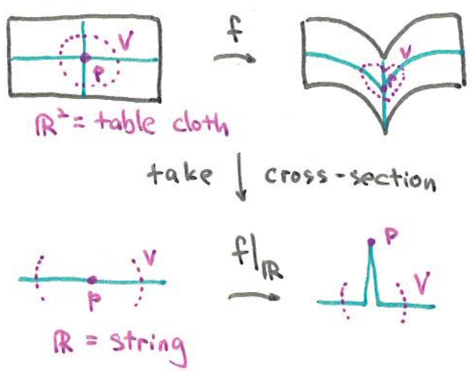


(0) shows when rank $(Df(p)) = 0$, an intuition comes from moving in a neighborhood of the peak (0), the projected position remains the same regardless the direction.

(1) shows rank $(Df(p)) = 1$, refers to moving in the up/down direction

changes the projected position, but not the other direction.

Say we want to show for a neighborhood V of $p \in C(1)$, we want to show there is regular value in V.



We can actually look for a regular value within the cross-section. Then the dimension goes from $2 \to 1$, and the rank of $Df(1)$ goes $2 \to 1$. The idea for the next steps is to show if $\operatorname{rank}(Df(p)) \neq 0$, we can reduce the dimension until it is.

Defn. The super-critical points of f is the set

$$C_f^{\#} := \{ p \in U \mid Df(p) = 0 \} \quad \Leftrightarrow \operatorname{rank}(Df(p)) = 0.$$

The remainder of the proof will consist of the next two steps.

Step 3. We will show $\mathbb{R}^n \setminus f(C_f^{\#} \cap A)$ is dense.

Step 4. We will show how to reduce the rest of U so that we can induct.

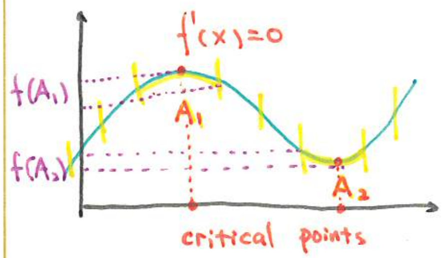
The idea of the rest of the proof is to deal with super-critical points, using a volume argument. Then will reducing other critical points to super-critical points by taking cross-sections. Step 4 will deal with all other regions of U. Now let's start with Step 3.

Step 3. Consider $A \subseteq \mathbb{R}^n$ a closed cube with side length ℓ. Then $\operatorname{vol} A = \ell^n$.

We will show the follow lemma:

Lemma. $\forall \varepsilon > 0$, $f(C_f^{\#} \cap A)$ can be covered by a finite number of cubes with total volume $< \varepsilon$.

Once this is proved, it will not be hard to show $\mathbb{R}^n \setminus f(C_f^{\#} \cap A)$ is dense.



sketch of the proof. Say A is partitioned into cubes (intervals for 1D) A_i, such that for $x, y \in A_i$, $|f'(x) - f'(y)| < \delta$ for intervals A_i that contain critical value(s). Then if $A_i \cap C_f^\# \neq 0$, then we know $|f'(x)| < \delta, \forall x \in A_i$ and $\frac{vol(f(A_i))}{Vol(A_i)} < \delta$.

Idea. Given the first derivative of the super-critical points being zero, the derivative of all points in the interval is bounded. The key idea is:

$$\text{Bound on } f' \xRightarrow{MVT} \text{Bound on range}(f)$$

How do we know such interval A_i exist?

A_i is chosen such that $x, y \in A_i$, $|f'(x) - f'(y)| < \delta$. This is possible because $f \in C^\infty$, so f' is C^∞ and Lipschitz. Say f' is Lipschitz such that $|f'(x) - f'(y)| < M|x-y|$. Then we can choose A_i s.t. $|A_i| = \frac{\delta}{M}$. So the size of the interval will depend on δ, but will not get too small.

1D case. Let $x^* = \underset{x \in A_i}{\arg\max} f(x)$, $y^* = \underset{y \in A_i}{\arg\min} f(y)$. So $f(A_i)$ is covered by an interval of length $f(x^*) - f(y^*)$.

By MVT, $f(x^*) - f(y^*) = f'(c)(x^* - y^*)$, $c \in A_i$

$$< \delta |x^* - y^*|$$

We only need to consider A_i such that $A_i \cap C_f^\# \neq \phi$. So there exists $y \in A_i$ such that $f'(y) = 0$. So $\forall x \in A_i$, $|f'(x) - 0| < \delta$.

As a result, $\frac{vol(f(A_i))}{vol(A_i)} < \delta$ for A_i containing super-critical point.

Full proof. We want to show for any $\varepsilon > 0$, we can cover the image of the super-critical points with finite number of cubes with total volumn $< \varepsilon$. We are doing this with closed cube A with side length ℓ.

proof. We choose $\delta > 0$, such that $\delta^n < \frac{\varepsilon}{\ell^n n^n}$ (reason will show up later)

There exists some N such that A can be divided into N^n subcubes of such length $\frac{\ell}{N}$ and

$$\boxed{\begin{array}{l}\text{for any } x, y \text{ in the same subcube} \\ \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| < \delta \quad \forall i, j\end{array}}$$

This is possible as $\frac{\partial f_i}{\partial x_j}$ is Lipchitz.

Label the subcubes which intersect $C_f^\#$ as $A_1, A_2, \dots, A_m$, $m \le N^n$.

We want to show $f(A_k)$ can be covered by a small cube.

Idea: Since A_k intersects $C_f^\#$, $\exists\, y \in A_k$ s.t. $\frac{\partial f_i}{\partial x_j}(y) = 0 \quad \forall i, j$.

$$\Rightarrow \forall x \in A_k, \ \left| \frac{\partial f_i}{\partial x_j}(x) \right| < \delta$$

Similar to the 1D case, we will use the bound on the derivative to correspond with the range of $f(A_k)$.

For any $x, y \in A_k$, $\exists\, c$ on the segment from x to y such that

by MVT: $f_i(x) - f_i(y) = \nabla f_i(c) \cdot (x - y)$,

where $f_i : \mathbb{R}^n \to \mathbb{R}$ denotes the ith coordinate of f.

Hence, $|f_i(x) - f_i(y)| = |\nabla f_i(c) \cdot (x - y)|$ each x, y contained in a subcube of side length $\frac{\ell}{N}$.

by triangle inequality: $\le \sum_{j=1}^{n} \underbrace{\left| \frac{\partial f_i}{\partial x_j}(c) \right|}_{< \delta \text{ from above}} \cdot |x_j - y_j| < n \delta \frac{\ell}{N}$.

In particular, for each coordinate i,

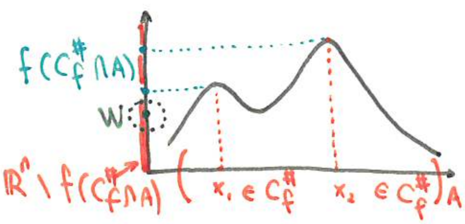
$$\max_{x \in A_k} f_i(x) - \min_{y \in A_k} f_i(y) \le n \delta \frac{\ell}{N}$$

Thus, $f(A_k)$ can be contained in a cube of side length $n \delta \frac{\ell}{N}$, so volumn $(n \delta \frac{\ell}{N})^n$. So aggregating this across all $m \le N^n$ subcubes, we can cover $f(C_f \cap A)$ with total volume $m \cdot \left(\frac{n \delta \ell}{N}\right)^n \le n^n \delta^n \ell^n < \varepsilon$. □

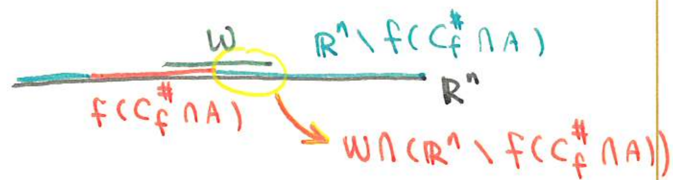
Now we will use this to show why the complement

$$\mathbb{R}^n \setminus f(C_f^{\#} \cap A)$$

is dense. In other words, for any point $p \in \mathbb{R}^n$ and neighborhood W of p, then $W \setminus f(C_f^{\#} \cap A)$ is non-empty. This is because:



$$W \cap (\mathbb{R}^n \setminus f(C_f^{\#} \cap A)) = W \setminus f(C_f^{\#} \cap A)$$



Suppose for the sake of contradiction that there is such a neighborhood W such that $W \setminus f(C_f^{\#} \cap A) = \phi$. Then consider cube $B \subset W$, so

$$B \subset W \subseteq f(C_f^{\#} \cap A)$$

since $W \setminus f(C_f^{\#} \cap A) = \phi$.

Let $\varepsilon = \mathrm{vol}(B)$, our lemma tells that $f(C_f^{\#} \cap A)$ can be covered by a finite number of cubes of total volume $< \varepsilon$. (W is an open neighborhood of p, so there exists a cube $B \subset W$ with none zero volume.) The idea is even we choose W really small, we can cover $f(C_f^{\#} \cap A)$ with an even smaller volume. So this contradiction brings $W \setminus f(C_f^{\#} \cap A) \neq \phi$. Therefore, we have proved $\mathbb{R}^n \setminus f(C_f^{\#} \cap A)$ is dense. □

<u>Step 4</u>. Induction on dimension.

Sard's Theorem. Let U be an open subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ a C^∞ map. Then $\mathbb{R}^n \setminus f(C_f)$ is dense in $\mathbb{R}^n$.

Baire category theorem will be applied here. We want to show $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense. We write $A = w_1 \cup w_2 \cup \cdots$. Then by De Morgan's law,

$$\begin{aligned} \mathbb{R}^n \setminus f(C_f \cap A) &= \mathbb{R}^n \setminus f(C_f \cap A \cap (\bigcup_i w_i)) = \mathbb{R}^n \setminus f(\bigcup_i (C_f \cap A \cap w_i)) \\ &= \mathbb{R}^n \setminus (\bigcup_i f(C_f \cap A \cap w_i)) = \bigcap_i (\mathbb{R}^n \setminus f(C_f \cap A \cap w_i)) \end{aligned}$$

By Baire category theorem, if we show $\mathbb{R}^n \setminus f(C_f \cap A \cap w_i)$ is dense, then $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense.

Goal of Step 4. We will show how to construct a new function:

$$f^{new} : U^{new} \to \mathbb{R}^{n-1}$$

from f, which will allow us to induct. The following proposition will allow us to compose f with diffeomorphisms.

Remark 3.7.4. Let $g : W \to U$ be a diffeomorphism, and

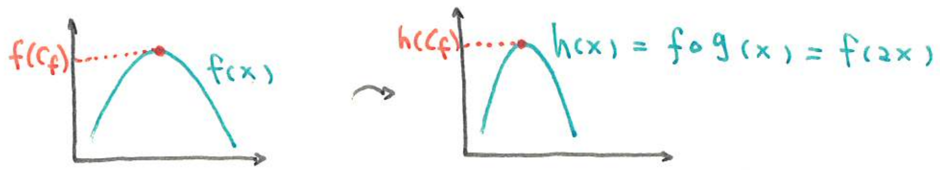
$$h := f \circ g, \quad h : W \to \mathbb{R}^n.$$

Then the set of critical values of f is the same as h:

$$f(C_f) = h(C_f).$$

So if we slip in a diffeomorphism into f, that won't change the regular values and the critical values. So Sard's theorem for h is equivalent to Sard's theorem for f.

Eg. Consider $g(x) = 2x$, and some $f(x)$ looks like:



proof of Remark 3.7.4. By the chain rule,

$$Dh(p) = Df(g(p))Dg(p)$$

$$\Rightarrow \det(Dh(p)) = \det(Df(g(p)))\det(Dg(p)).$$

Since g is a diffeomorphism, $\det(Dg(p)) \neq 0$.

Hence, for $p \in W$, $\det(Dh(p)) = 0 \Leftrightarrow \det(Df(g(p))) = 0$

This says, if p is a critical point of h, i.e., $p \in C_h$, then $g(p)$ is a critical point of f, i.e., $g(p) \in C_f$ and vice versa.

So $g(C_h) = C_f \Rightarrow h(C_h) = f\circ g(C_h) = f(C_f)$. □

proof of Step 4. Now we complete the proof that $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense.

We will use induction on dimension.

Base case: $n=1$, $C_f = C_f^\#$, proved in Step 3.

Hypothesis (Sard's theorem). For any C^∞ map $f: U \to \mathbb{R}^n$, U is an open subset of $\mathbb{R}^n$, we have $\mathbb{R}^n \setminus f(C_f)$ is dense in $\mathbb{R}^n$.

More specifically, Step 3 shows $\mathbb{R}^n \setminus f(C_f^\# \cap A)$ is dense for a compact set A, and Step 1 shows proving $\mathbb{R}^n \setminus f(C_f \cap A)$ is dense, suffice to show $\mathbb{R}^n \setminus f(C_f)$ is dense. This generalize to $C_f^\#$ since the proof involves BCT (Remark 3.7.2) and $\mathbb{R}^n \setminus f(C_f \cap A)$ being open (Remark 3.7.3), and both apply to the $C_f^\#$ case. So proving $\mathbb{R}^n \setminus f(C_f^\# \cap A)$ is dense suffice for the $C_f^\#$ case also.

Inductive step: We first provide a few definitions, followed by an outline:

- Let $U_{i,j}$ be the open subset of U where $\frac{\partial f_i}{\partial x_j} \neq 0$.
- $g: U_{1,1} \to \mathbb{R}^n$ $\qquad f = (f_1, \dots, f_n)$

  $(x_1, \dots, x_n) \mapsto (f_1(x), x_2, \dots, x_n)$ $\qquad f(x) = (f_1(x), \dots, f_n(x))$
- $h: W_i \longrightarrow \mathbb{R}^n$, $\quad W_i := g(V_i)$, where $A \subset \bigcup_i V_i$

  $h = f \circ g^{-1}$, $\quad h(x) = (h_1(x), \dots, h_n(x))$.
- $\tau: \mathbb{R}^n \to \mathbb{R}^{n-1}$

  $(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n)$

Outline: 1. We argue that we can focus on $U_{1,1} \subseteq U$.

2. We construct h with $h(C_h) = f(C_f)$, $h_1(x) = x_1$.

3. To prove $\mathbb{R}^n \setminus h(C_h)$ is dense, consider $p \in \mathbb{R}^n$ and neighborhood W. Say $p = (c, \tau(p))$, we want to show W intersects $\mathbb{R}^n \setminus h(C_h)$. Define $W_c := \{\tau(x) \mid x_1 = c, x \in W\} \subseteq \mathbb{R}^{n-1}$, $h_c := \tau(h(c, \dots))$, $h_c: W_c \to \mathbb{R}^{n-1}$. We will show $q \in h_c(C_{h_c}) \Rightarrow (c, q) \in h(C_h)$ (dim $n-1$; dim n)

4. Induction: $W_c \cap (\mathbb{R}^{n-1} \setminus h_c(C_{h_c}))$ non-empty $\Rightarrow W \cap (\mathbb{R}^n \setminus h(C_h))$ non empty

Note that $U = C_f^{\#} \cup \big(\bigcup_{1 \le i,j \le n} U_{i,j} \big)$, appealing to De Morgan's law:

no entry is non-zero

some entry is non-zero

$$\underbrace{\mathbb{R}^n \setminus f(C_f \cap A)}_{} = \mathbb{R}^n \setminus f(C_f \cap A \cap U)$$

Goal is to show this set is dense

$$= \mathbb{R}^n \setminus f\Big(C_f \cap A \cap \big(C_f^{\#} \cup \bigcup_{1 \le i,j \le n} U_{i,j}\big)\Big)$$

$$= \mathbb{R}^n \setminus f\Big(\big(\underbrace{C_f \cap A \cap C_f^{\#}}_{C_f^{\#} \cap A}\big) \cup \big(C_f \cap A \cap \big(\bigcup_{1 \le i,j \le n} U_{i,j}\big)\big)\Big)$$

$$= \mathbb{R}^n \setminus f\Big((C_f^{\#} \cap A) \cup \Big(\bigcup_{1 \le i,j \le n} (C_f \cap A \cap U_{i,j})\Big)\Big)$$

De Morgan's law | De Morgan | De Morgan

$$= \big(\mathbb{R}^n \setminus f(C_f^{\#} \cap A)\big) \cap \Big(\mathbb{R}^n \setminus f\Big(\bigcup_{1 \le i,j \le n} (C_f \cap A \cap U_{i,j})\Big)\Big)$$

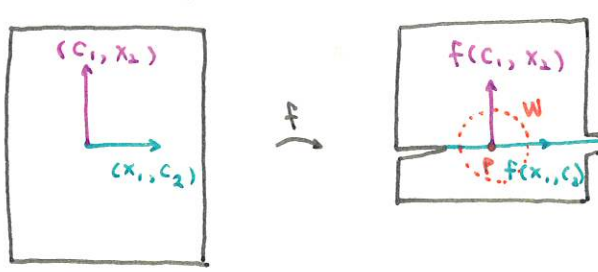
De Morgan's law

$$= \underbrace{\big(\mathbb{R}^n \setminus f(C_f^{\#} \cap A)\big)}_{\text{We have proved this set is dense in Step 3}} \cap \underbrace{\Big(\bigcap_{1 \le i,j \le n} \mathbb{R}^n \setminus f(C_f \cap A \cap U_{i,j})\Big)}_{\text{We need to show this set is dense, then by BCT, LHS is dense}}$$

So we can focus on $f : U_{i,j} \to \mathbb{R}^n$. WLOG, we take $i = j = 1$. so

$$\frac{\partial f_1}{\partial x_1}(p) \ne 0 \quad \forall p \in U_{1,1}.$$

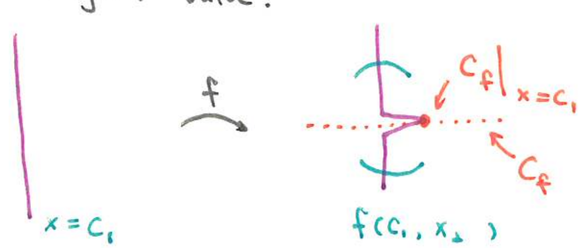
Diagram for $\frac{\partial f_1}{\partial x_1} \ne 0$.



$f(x,y) = (x, \text{piecewise in } y)$

If we want to show a neighborhood W of p intersects $\mathbb{R}^n \setminus f(C_f)$, we reduce to a section in the y-direction.

So we take a section with $x = c_1$, then if $W \cap (\mathbb{R}^n \setminus f(C_f))$, $W|_{x=c_1}$ still have a regular value.



i.e., there is still regular value in this slice.

So given $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, our goal will thus be to obtain $h(x_1, x_2) = (x_1, f_2(*, x_2))$ in 2D. In general:

$$h(x_1, \dots, x_n) = (x_1, f_2(*, x_2, \dots, x_n), \dots, f_n(*, x_2, \dots, x_n)).$$

The goal is to cut off f_1 and replace with x_1 somehow. Recall we defined

$$g : U_{1,1} \to \mathbb{R}^n$$

$$(x_1, \dots, x_n) \mapsto (f_1(x), x_2, \dots, x_n)$$

$$Dg = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \Rightarrow \det(Dg) = \frac{\partial f_1}{\partial x_1} \neq 0$$

since $g : U_{1,1} \to \mathbb{R}^n$.

We consider this g because $h := f \circ g^{-1}$, we will get the h we desire.

By the inverse function theorem, g is locally a diffeomorphism at point $p \in U_{1,1}$.

So A can be covered by a finite number of open sets $V_1, \dots, V_m$ so that g is a diffeomorphism on each V_i, and $W_i := g(V_i)$ is open.

Because $A \subseteq V_1 \cup \cdots \cup V_m$, we have

$$\mathbb{R}^n \setminus f(C_f \cap A) = \bigcap_i \mathbb{R}^n \setminus f(C_f \cap A \cap U_i)$$ Page 48

So again by BCT, suffice to show $\mathbb{R}^n \setminus f(C_f \cap A \cap U_i)$ is dense.

Now we have g is a diffeomorphism on each $V_i \to W_i \subset \mathbb{R}^n$, and

g^{-1} is a diffeomorphism on each $W_i \to V_i$, now define

$h := f \circ g^{-1}$, $W_i \to \mathbb{R}^n$.

Previously, we have showed $f(C_f) = h(C_f)$. So we can focus on proving $\mathbb{R}^n \setminus h(C_h)$ is dense.

$$V_i \xrightarrow{g} W_i$$

Q. What is $h(x_1, \dots, x_n)$?

A. Say $x_1 = f_1(x_0, x_2, \dots, x_n)$. We know $\exists$ such x_0 because $h : W_i \to \mathbb{R}^n$

So $(x_1, \dots, x_n) \in W_i$, so $x_1 = f_1(x_0, x_2, \dots, x_n)$ by definition of g.

Then $f \circ g^{-1}(x_1, \dots, x_n) = f(x_0, x_2, \dots, x_n)$

$$= (f_1(x_0, x_2, \dots, x_n), \dots, f_n(x_0, x_2, \dots, x_n))$$

$$= (x_1, f_2(x_0, x_2, \dots, x_n), \dots, f_n(x_0, x_2, \dots, x_n))$$

So $h = f \circ g^{-1}$ fixes the first coordinate, as desired.

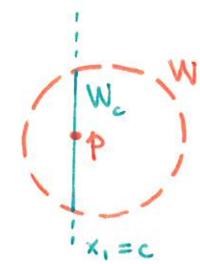
Now we show $\mathbb{R}^n \setminus h(C_h)$ is dense with inductive hypothesis for $n-1$.

We consider any point $p \in \mathbb{R}^n$ and say we wish to show that $\mathbb{R}^n \setminus h(C_h)$ intersects a neighborhood W around p. Recall that we defined

$$\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$$
$$(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n)$$

Write $p = (c, \tau(p))$ where $c \in \mathbb{R}$. Then let

$$W_c := \{\tau(x) \mid x_1 = c,\ x \in W\} \subseteq \mathbb{R}^{n-1}$$
$$h_c : W_c \longrightarrow \mathbb{R}^{n-1}$$
$$x \mapsto \tau(h(c, x))$$



By inductive hypothesis for dim $n-1$, we know $\mathbb{R}^{n-1} \setminus h_c(C_{h_c})$ is dense.

<u>Idea.</u> The idea is to show that for $q \in \mathbb{R}^{n-1}$ such that

$$q \in (\mathbb{R}^{n-1} \setminus h_c(C_{h_c})) \cap W_c.$$

it can be expanded to (c, q) and $(c, q) \in (\mathbb{R}^n \setminus h(C_h)) \cap W$.

To prove this, we first show

$$\det(Dh_c(x_2, \dots, x_n)|_{(x_2, \dots, x_n)}) = \det(Dh(x_1, \dots, x_n)|_{(c, x_2, \dots, x_n)})$$

$$Dh(x_1, x_2, \dots, x_n) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \cdots 0 \\ \frac{\partial h_2}{\partial x_1} & \\ \vdots & \boxed{Dh_c(x)} \\ \frac{\partial h_n}{\partial x_1} & \end{bmatrix}$$

evaluated at $(c, x_2, \dots, x_n)$ — since $h_1(x) = x_1$

evaluated at $(x_2, \dots, x_n)$

This shows $\det(Dh(x_1, \dots, x_n)) = \det(Dh_c(x_2, \dots, x_n))$ via Laplace expansion. This also shows that if $(x_2, \dots, x_n) \in C_{h_c}$, $(c, x_1, \dots, x_n) \in C_h$.

Also, if $(c, x_2, x_3, \dots, x_n) \in C_h$, $(x_2, \dots, x_n) \in C_{h_c}$. So

$$\begin{array}{ccc} x \in C_{h_c} & \Longleftrightarrow & x \in C_h \\ \Updownarrow & & \Updownarrow \\ \boxed{h_c(x) \in h_c(C_{h_c})} & & h(c,x) \in h(C_h) \\ & & \Updownarrow \\ & & \boxed{(c, h_c(x)) \in h(C_h)} \end{array}$$

This implies that $q = h_c(x) \in h_c(C_{h_c}) \Longleftrightarrow (c, q) \in h(C_h)$, as desired.
So the induction step concludes, since we have showed

$$q \in (\mathbb{R}^{n-1} \setminus h_c(C_{h_c})) \cap W_c$$

can be expanded to

$$(c, q) \in (\mathbb{R}^n \setminus h(C_h)) \cap W.$$

Assuming $\mathbb{R}^{n-1} \setminus h_c(C_{h_c})$ is dense by induction hypothesis, we can conclude that

$$(\mathbb{R}^{n-1} \setminus h_c(C_{h_c})) \cap W_c$$

is non-empty, so

$$(\mathbb{R}^n \setminus h(C_h)) \cap W \cap (x_1 = c)$$

is non-empty. Therefore,

$$(\mathbb{R}^n \setminus h(C_h)) \cap W$$

is non-empty, i.e., $\mathbb{R}^n \setminus h(C_h)$ is dense. $\square$